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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.

2. To supply an additional medium for the publication of expository mathematical articles.

3. To promote more scientific methods ot teaching mathematics.

4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Mathematics---Human Laboratory Instrument

If by magic or other means there could be created at the end of the present school year a nation-wide situation in which would be found recorded in no school or college a single failure in mathematics, the present tide of popular opposition to mathematics would recede with startling swiftness. In lieu of the present annual outpouring from our schools of thousands of young people whose hostility to the subject is born of repeated and humiliating failure in it, would be multitudes voicing their praise and admiration for it.

Under present conditions complete materialization of this ideal is little more than a dream. But it may not be unreasonable to assume that a partial realization of the dream could be brought about by the adoption of a new definition of the *fundamental AIM of mathematics teaching in every mathematics classroom of the land.*

The definition would assign paramount importance to the *human subject,* while mathematics and its study would be invested with the role of mere *instruments* of *service* in the *cause* of the *student's orderly mental development.* Conforming to the new orientation, the lecturer in mathematics would largely be replaced by the intimate counsellor with the student. The dominating inquiries would be in the direction of effective modes of stimulating the student to the discovery of his own powers and weaknesses through mathematical experience. The human laboratory viewpoint would control in every mathematics class, with mathematics the only laboratory instrument.

All programs would postulate the ability of every normal mind to consciously and thoroughly experience a certain amount of mathematics. OUTLAWED would be such declarations as the following:

*"There are some (students) who are unwilling and *unable* to think logically. . . . They do not have the most important characteristic distinguishing them from the other members of the animal kingdom." . . . "I would hate to have to teach mathematics to *all* stu-

*Quotations from the returns of the Seidlin questionnaire. See Vol. XI, No. 1, National Mathematics Magazine.

dents of a liberal arts college. ... I do not care what they do with the morons." ... "Mathematics is meat for strong men, not milk for babes and sucklings. Let us frankly admit it and not seek the unfit." ... "I have in mind the type of person who has real ability in other lines, but who has a blind spot in the direction of mathematics." The public stigmatizing of certain minds as mathematically impotent merely on the evidence of a few oral or written quizzes would be *indicted* by this new orientation as *unscientific*, if not immoral. Upon such slender evidence Isaac Newton even in his later 'teens might have been invited to abandon mathematics because he seemed to like water-clocks and windmills so much more than mathematics.

A fundamental implication of the new teaching objective would be programs aiming at the interest motivation of each student. The so-called brilliant type would require different interest motivation from the slow-moving type. If, initially, the former type should, by nature, or by his previous training, possess both interest and ability in mathematics, the problem of his interest motivation would already largely be solved. Mathematical development for him could with safety be left to the stimulating influences of: first, the consciousness of success in mathematics; second, acknowledgment of that success by his fellows; third, the intellectual pleasure springing out of a growing understanding of mathematics; fourth, periodic contacts with his instructor; fifth, the stimulus of competition with others at the same level of ability and interest. Obviously, the individuals of this (brilliant) type could from time to time as need should arise contact the instructor as a group rather than as separate individuals, since demand for separate individual treatment presumably would be slight.

The type of individual described as having mathematical ability without special mathematical interest would, at first, be subject for individual treatment. Interest-motivation for him should, with little difficulty on the part of the wise instructor, be rooted in the fact that his ability has already been recognized by others. In time, this type would under proper stimulation merge into the interest-and-ability type, and would then be treated similarly to that type.

It is apparent that by far the largest amount of planning and effort done in this mathematical human laboratory project would be expended in motivating mathematical interest, (1), of the slow moving

type, (2), of the type assumed to be wholly deficient in mathematical ability. In view of the postulation of the paramount importance of the individual human subject in this new philosophy of mathematics teaching, any occasional or even periodic group treatment of the two last named types would be subordinated to separate programs nucleated about the individual *as individual.* Small groups of those individuals that experience similar or nearly similar difficulties in mathematics could with advantage receive group treatment but, largely, only upon the free initiative of the individuals. Such group treatment, in a great many cases, could with advantage take the form of a discussion of the difficulties by the group rather than the form of a ready-made solution of the difficulties by an instructor.

Though our description of all the immediate logical implications of the new DEFINITION of mathematics teaching is by no means complete, lack of space requires the present closing of our study. Our final paragraph shall consist of two statements which in our judgment have a significance not be overestimated.

(1) Professor Glenn James of the University of California at Los Angeles has furnished the world with near-complete statistical evidence showing that this human laboratory viewpoint of mathematics is justified, by results covering a period of many years in which the method was subjected by him to most careful experimentation. Some of the evidence was published in this journal under the title *The Cause and Cure of Delinquency in Freshman Mathematics.* His report appeared in the March 1937 issue of National Mathematics Magazine. (2) The Glenn James experiments alluded to in the first statement appear to check with the following hypothesis: *Mathematics being in its last analysis a type of rigorous but normal thinking should be subject to mastery as a method of thinking by every mind not ABNORMAL or SUB-NORMAL in character, i. e., by ANY NORMALLY CONSTITUTED intelligence.*

S. T. SANDERS.

GEORGE A. MILLER

VOLUME XII, No. 8 OF THE NATIONAL MATHEMATICS MAGAZINE IS DEDICATED TO PROFESSOR GEORGE A. MILLER AND PRESENTED TO HIM ON THE OCCASION OF HIS SEVENTY-FIFTH BIRTHDAY ANNIVERSARY JULY 31, 1938.

A General Theory of Limits

By H. L. SMITH
Louisiana State University

In 1922 E. H. Moore and the author* published a general theory of limits which had as instances nearly all known examples of limits of numerically valued functions. There was one important limit not covered by the theory, the so-called approximate limit used in the advanced theory of integration. It is the purpose of the present note to present the foundation of a new general theory which includes the older theory and also the approximate limit referred to above. In order to reach as large a group of readers as possible, we shall make our exposition independent of the paper cited above and also rather full.

CERTAIN CLASSICAL INSTANCES COVERED BY THE THEORY. As the definition of limit to be given below is quite abstract, the reader will do well to keep in mind certain classical limit definitions which are covered. We list these.

Definition 1. A sequence $\{a_n\}$ of numbers converges to a number a as limit, in notation, $\lim\limits_{n\to\infty} a_n = a$, provided for every positive number e there is an integer n_e (dependent on e) such that $|a-a_n| \leqq e$ for every $n \geqq n_e$.

Definition 2. A function $f(x)$ converges to b as x approaches a, in notation, $\lim\limits_{x\to\infty} f(x) = b$, provided for every positive number e there is a positive number d_e such that $|b-f(x)| \leqq e$ for every x such that $0 < |x-a| \leqq d_e$.

Definition 3. Let D be any partition of the interval $a \leqq x \leqq b$ into sub-intervals $x_{i-1} \leqq x \leqq x_i$ $(i = 1, \cdots, n)$ such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We call length of the longest of the sub-intervals $x_{i-1} \leqq x \leqq x_i$ the norm of the partition D. Then any function $f(x)$ defined for $a \leqq x \leqq b$ gives rise to a (multiply-valued) function of D which we denote by $f(D)$ and define as follows:

$$f(D) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

where

$$x_{i-1} \leqq \xi_i \leqq x_i \qquad (i = 1, \cdots, n).$$

*Am. Journal of Mathematics, Vol. XLIX, (1922).

We then say that $f(D)$ converges to I as the norm of D approaches zero provided to every positive e there corresponds a positive d_e such that

$$|I-f(D)| \leqq e$$

for every D of norm $\leqq d_e$. The limit I when it exists, is, of course, the Riemann integral of $f(x)$ between the limits a and b.

Definition 4. Let $f(x)$ be a measurable function on some interval containing a. We then say $f(x)$ converges approximately to b as x approaches a, in notation, $\lim\limits_{x\to\infty} \text{ap}\, f(x)=b$, provided the class of all values of x different from a and such that

$$|b-f(x)| \leqq e$$

has a as point of unit density, for every positive value of e.

RESTATEMENT OF DEFINITIONS 2 AND 3. These four definitions seem at first sight to be quite different. We shall therefore restate Definitions 2 and 3 in forms much more like that of Definition 1.

Definition 2′. Let X be the set of all numbers distinct from a. Then $\lim\limits_{x\to\infty} f(x)=b$, provided to every e there is an x_e in X such that

$$|b-f(x)| \leqq e$$

for every x in X such that $|x-a| \leqq |x_e-a|$.

It is very easily shown that this definition is equivalent to Definition 2.

Definition 3′. If $D, f(x), f(D)$ are as in Definition 3 and if we say that a partition D_1 of $a \leqq x \leqq b$ is finer than a partition D_2 of that interval whenever every one of the intervals of which D_1 is made up is a subinterval of some interval of D_2, then $f(D)$ converges to I provided for every e there is a partition D_e such that

$$|I-f(D)| \leqq e$$

for every partition D finer than D_e.

THE MOORE-SMITH LIMIT. The Definitions 1, 2′, 3′, given above are now seen to be in close analogy; in each case we have a numerically-valued function defined for every element of a certain class, and in each case the function converges to a numerical limit in case an inequality involving an arbitrary positive number e is satisfied for all elements of the class which are in a certain relation to a certain element of the class which depends on e. This suggests the possibility of an abstract theory of which these definitions are all instances, and even

suggests the form of that theory. We shall need to postulate a class $\mathfrak{P}$ of elements p and a relation R between certain pairs of elements of $\mathfrak{P}$. That is, we assume that if p_1, p_2 are a pair of elements of $\mathfrak{P}$, then we know whether p_1 is, or is not, in the R relation to p_2. We express the fact that p_1 is in the R relation to p_2 by the notation p_1Rp_2. On this basis we now state

Definition 5. If $\varphi(p)$ is a numerically valued function defined for every p of $\mathfrak{P}$, then $\varphi(p)$ converges to a, in notation, $\lim \varphi(p)=a$, provided for every positive e there is an element p_e of $\mathfrak{P}$ such that

$$|a-\varphi(p)| \leqq e$$

for every p of $\mathfrak{P}$ in the R relation to p_e.

This is the definition of limit given by E. H. Moore and the author in the paper cited above. In order to be able to develop a theory which would cover the principal results known in the cases of Definitions 1, 2, 3 it was necessary to assume that R satisfied certain postulates. These are as follows:

(1) R is transitive, that is, if p_1Rp_2 and p_2Rp_3 then p_1Rp_3.

(2) R has the composition property, that is, if p_1,p_2 are any elements in $\mathfrak{P}$, then there is an element p_{12} of $\mathfrak{P}$ such that $p_{12}Rp_1$, $p_{12}Rp_2$.

The theory developed on this basis is quite simple and, as indicated above, covers nearly all important cases. It does not however cover *Definition 4* without a somewhat artificial transformation. We therefore present a new theory which does include all four of the definitions given above as instances.

ANOTHER FORMULATION OF DEFINITIONS 1, 2, 3. Before stating our new abstract definition of limit, we restate Definitions 1, 2, 3 in forms similar to that of Definition 4.

Definition 1'. A sequence $\{a_n\}$ converges to a number a if the set of all values of n for which

$$|a-a_n| \leqq e$$

contains all integers greater than a certain one.

Definition 2''. A function $f(x)$ converges to a number b as x converges to a if for every positive number e the set of all values of x for which

$$|b-f(x)| \leqq e$$

contains all points except a of a certain interval having a as centre.

Definition 3″. If D, $f(x)$, $f(D)$ are as in Definition 3 then $f(D)$ converges to I if for every e the class of all partitions D of $a \leqq x \leqq b$ such that

$$|I - f(D)| \leqq e$$

contains all partitions whose norms are less than a certain number.

Definition 3‴. If D, $f(x)$, $f(D)$ are as in Definition 3 then $f(D)$ converges to I if for every e the class of all partitions D such that

$$|I - f(D)| \leqq e$$

contains every partition finer than a certain one.

The reader will easily verify the fact that 1′ is equivalent to 1; that 2″ is equivalent to 2; and that 3″, 3‴ are each equivalent to 3. It is also clear that each of these definitions is in close analogy with 4. Thus in each case a numerically valued function defined for every element of a certain class converges to a number as a limit provided for every positive e the class of all elements for which a certain inequality involving e is satisfied has a certain property. This suggests the abstract definition which we are about to give.

THE NEW ABSTRACT DEFINITION OF LIMIT. As a basis for our new definition we assume that we have a class $\mathfrak{P}$ of elements p and a property C possessed by certain subsets S of $\mathfrak{P}$. We then have

Definition 6. A numerically valued function $\varphi(p)$ defined on $\mathfrak{P}$ converges to a number a if for every positive e the class of all elements p such that

$$|a - \varphi(p)| \leqq e$$

has the property C.

In order to secure an adequate body of theory on the basis of this definition, we assume that the property C satisfies the following postulates.

I. If a set S has the property C it contains at least one element.

II. If S_1 and S_2 both have the property C then the set $S_1 \cdot S_2$ of all points which S_1 and S_2 have in common also has the property C.

III. If S_1 has the property C and S_2 includes S_1 as a subset, then S_2 has the property C.

We now show that *Definition 5* is a special case of *Definition 6*; that is, we show that if $\mathfrak{P}$ and R are as above, then we can define in terms of R a property $C(R)$ which satisfies the postulates I, II, II and is also such that if $\lim \varphi = a$ in the sense of *Definition 5* then also $\lim \varphi = a$

in the sense of Definition 6. To do this we have only to define the property $C(R)$ as follows. A set S has the property $C(R)$ if, and only if, it contains every point p in the R relation to a certain point of it. We leave to the reader the proof that this does indeed suffice.

ELEMENTARY PROPERTIES OF LIMITS. The following elementary theorems are true:

1. If $\lim \varphi = a$, $\lim \varphi = b$, then $a = b$.
2. If $\lim \varphi_1 = a_1$, $\lim \varphi_2 = a_2$, then $\lim (\varphi_1 + \varphi_1) = a_1 + a_2$ and $\lim \varphi_1\varphi_2 = a_1a_2$.
3. If $\lim \varphi = a \neq 0$, then $\lim \dfrac{1}{\varphi} = \dfrac{1}{a}$.

To prove these we employ the notation $E(|a-\varphi| \leqq e)$ for the set of all points p such that $|a - \varphi(p)| \leqq e$.

Proof of 1. By hypothesis each of the classes $E(|a-\varphi| \leqq e)$, $E(|b-\varphi| \leqq e)$ has the property C; hence their product

$$E(|a-\varphi| \leqq e) \cdot E(|b-\varphi| \leqq e),$$

that is, the set of all the points that they have in common, has the property C. Let p_e be on a point in the product. Then we have

$$a - e \leqq \varphi(p_e) \leqq a + e,$$

$$b - e \leqq \varphi(p_e) \leqq b + e.$$

Hence $$a - e \leqq b + e, \quad b - e \geqq a + e,$$

or $$a - b \leqq 2e, \quad b - a \leqq 2e.$$

Since this is true for every positive number e, we have $a = b$, as was to be proved.

Proof of 2. The first part of 2 follows from the inequality

$$E(|a_1 - \varphi_1| \leqq e/2) \cdot E(|a_2 - \varphi_2| \leqq e/2) \subseteqq E(|(a_1 + a_2) - (\varphi_1 + \varphi_2)| \leqq e),$$

while the second part follows from the following

$$E(|a_1 - \varphi_1| \leqq e) \cdot E(|a_2 - \varphi_2| \leqq e) \subseteqq E(|a_1a_2 - \varphi_1\varphi_2| \leqq [|a_1| + |a_2| + e]e).$$

NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A LIMIT. We have the following result.

1. In order that $\lim \varphi = a$, it is necessary and sufficient that there exist a set S_e which has the property C and is such that

$$|a - \varphi(p)| \leqq e$$

for every point p in S_e.

The necessity is shown by taking S_e to be the set $E(|a-\varphi|\leqq e)$; the sufficiency follows from the inequality $S_e \subseteqq E(|a-\varphi|\leqq e)$ and III.

2. In order that lim φ exist, it is necessary and sufficient that there exist a set S_e with the property C and such that

$$|\varphi(p_1)-\varphi(p_2)|\leqq e$$

for every pair p_1p_2 such that both belong to S_e.

To prove the necessity we have only to take S_e to be the set $E(|a-\varphi|\leqq e/2)$, where a is the value of the limit. To prove the sufficiency take T_n to be S_e for $e=1/2^n$. Then take p_n in the set $T_1T_2\cdots T_n$, which has the property C by II and therefore is not empty by I. Then the sequence $\{p_n\}$ satisfies the inequality

$$|\varphi(p_m)-\varphi(p_n)|\leqq 1/2^n \quad (m \leqq n)$$

Hence $\varphi(p_n)$ converges to a number, say a, in the sense of *Definition 1*. But

$$\begin{aligned} T_1\cdots T_n &\subseteqq E(|a-\varphi|\leqq|a-\varphi(p_n)|+2^{-n}) \\ &\subseteqq E(|a-\varphi|\leqq e) \end{aligned}$$

for $n=n_e$, properly chosen. This completes the proof, by III.

IDEAL LIMITS. It is convenient to have for use in the sequel the notion of ideal limit. This we now define. We say that φ converges to $+\infty$, in notation, lim $\varphi=+\infty$, provided for every e the set $E(\varphi\geqq e)$ has the property C. If in this definition $\geqq e$ is replaced by $\leqq -e$ we have a definition of convergence of φ to $-\infty$.

MONOTONIC FUNCTIONS. A function φ is said to be monotonic increasing (relative to C) if for every point p_0 the set $E(\varphi\geqq\varphi(p_0))$ has the property C; it is said to be monotonic decreasing (relative to C) if the set $E(\varphi\leqq\varphi(p_0))$ has the property C for every p_0.

If φ is monotonic increasing, then lim φ exists, finite or infinite, and equals the least upper bound of φ.

We prove this for the case in which the least upper bound of φ is a number a. We have

$$\begin{aligned} E(\varphi\geqq\varphi(p_0)) &\subseteqq E(|a-\varphi|\leqq|a-\varphi(p_0)|) \\ &\subseteqq E(|a-\varphi|\leqq e) \end{aligned}$$

on choosing p_0 so that $|a-\varphi(p_0)|\leqq e$.

UPPER AND LOWER LIMITS. We now define lim sup φ to be the greatest lower bound of all a, finite or infinite, such that $E(\varphi\leqq a)$

has the property C and define lim inf φ to be the least upper bound of all a finite or infinite, such that $E(\varphi \geqq a)$ has the property C.

1. lim inf $\varphi \leqq$ lim sup φ.

For, set $M = \limsup \varphi$, $m = \liminf \varphi$. Then the sets $E(\varphi \leqq M+e)$, $E(\varphi \geqq m-e)$ both have the property C. Hence, by II, they have a common element. Hence

$$m-e \leqq M+e$$

for every positive e, so that $m = M$, as was to be proved.

2. For every function φ, lim sup φ is the least upper bound of all a such that $E(\varphi \leqq a)$ does not have the property C.

To prove this for lim sup φ finite we note that if $a_1 \leqq a_2$ then $E(\varphi \leqq a_1) \subseteqq E(\varphi \leqq a_2)$. Hence every number a_1 such that $E(\varphi \leqq a_1)$ does not have the property C is less than every number a_2 such that $E(\varphi \leqq a_2)$ does have the property C. Thus the class of all such numbers a_1 and the class of all such a_2 form a Dedekind cut in the real number system and to this cut corresponds the number lim sup φ.

3. In order that lim sup $\varphi = a$, it is necessary and sufficient that the following pair of conditions be satisfied:

1) there exists for every positive number e a set S_e with the property C and such that

$$\varphi(p) \leqq a+e$$

for every p in S_e.

2) if S is any set with the property C and e is any positive number, then there exists a point $p(S,e)$ in S such that

$$\varphi(p(S,e)) > a-e.$$

To prove the necessity, we take S_e to be the set $E(\varphi \leqq a+e)$, which has the property C by definition of lim sup φ, and hence is effective as the set S_e of 1). Also $E(\varphi \leqq a-e)$ does not have the property C. Hence since S does have that property, it follows from III that S must contain at least one point which does not lie in $E(\varphi \leqq a-e)$. Take $p(S,e)$ to be such a point. Then since $p(S,e)$ does not lie in $E(\varphi \leqq a-e)$ we have $\varphi(p(S,e)) > a-e$, which proves 2).

To prove the sufficiency, we note that by 1) and III the inequality

$$S_e \subseteqq E(\varphi \leqq a+e)$$

implies that the set $E(\varphi \geqq a+e)$ has the property C for every positive number e. Hence

$$\limsup \varphi \leqq a+e$$

for every positive e, so that

$$\text{(1)} \qquad \limsup \varphi \leqq a.$$

By 2) the set $E(\varphi \leqq a-e)$ does not have the property C since it does not contain any point at which $\varphi > a-e$. Hence

$$\limsup \varphi \geqq a-e$$

for every positive e, so that

$$\text{(2)} \qquad \limsup \varphi \geqq a.$$

From (1), (2) follows our result.

AN IMPORTANT SPECIAL CASE. We say that a class $\mathfrak{S}$ of subsets S of $\mathfrak{P}$ each of which has the property C has the property C if it contains an element S_0 such that every subset of S_0 having the property C belongs to $\mathfrak{S}$. Clearly the property C so defined for classes $\mathfrak{S}$ satisfies the postulates I, II, III, and therefore if $\varphi(S)$ is a numerically valued function defined for all sets S which have the property C, then *Definition 6* assigns a meaning to the symbol $\lim \varphi(S)$. We leave to the reader the proof of

1. In order that $\lim \varphi(S) = a$ it is necessary and sufficient that there be a set S_e with the property C such that

$$|a-\varphi(S)| \leqq e$$

for every S with property C which is a subset of S_e.

Now let $\varphi^*(S)$ denote the least upper bound of $\varphi(p)$ as p runs over S and let $\varphi_*(S)$ denote the greatest lower bound of $\varphi(p)$ as p runs over S. We then state

2. $$\limsup \varphi = \lim \varphi^*(S) = \text{l.bd. } \varphi^*(S).$$
$$\liminf \varphi = \lim \varphi_*(S) = \text{u.bd. } \varphi_*(S).$$

To prove the first of these, set $a = \limsup \varphi$, $b = \lim \varphi^*(S)$. Then $E(\varphi \leqq a+e)$ has the property C. Also

$$\varphi^*(E(\varphi \leqq a+e)) \leqq a+e.$$

But $$\varphi^*(E(\varphi \leqq a+e)) \geqq b$$

Therefore $$b \leqq a+e$$

for every e, so that

$$\text{(1)} \qquad b \leqq a.$$

Now take S_e so that S_e has the property C and $\varphi^*(S_e) \leqq b+e$. Then

$$S_e \subseteq E(\varphi \leqq b+e)$$

Hence $E(\varphi \leqq b+e)$ has the property C. Hence

$$b+e \geqq a$$

for every positive e, so that

$$b \geqq a. \tag{2}$$

From (1), (2) the result follows.

CORRIGENDA

Vol. XII

Page 47, line 22, for "*T. Mahrenholz*" read "*J. Mahrenholz*"; same page 49, line 5; page 50, line 18; page 51, line 7.
Page 54, line 14, for "*ABCL* "read "*ABC,L*".
Page 141, line 15, insert + between α^8 and $14\alpha^4\beta^4$.
Page 144, Fig. 1 belongs with No. 163, page 145.
Page 144, line 18, insert semicolon between ∞ and 2.
Page 146, line 9, for $2a/(a^2-1)$ read $2a^2/(a-1)$.
Page 148, line 22, the explanatory phrase "have fallen 'tails' in every trial" should be enclosed in parentheses. Line 26, for j substitute k.
Page 149, line 5, for the quantity (2^p-2^2) read (2^p-2); line 17, omit No. 125.
Page 151, bottom line, omit "of the".
Page 194, line 20, for K' read k'.
Page 200, line 14, for No. 104 read No. 204.
Page 304, invert figure. Concerning lines 1-4: The March issue of the American Mathematical Monthly has since appeared with the solution of E 291, showing that 19569^2 is *not* unique, but that 29106^2 or 847159236 is divisble by 99 and hence also by 66.
Page 307, line 15, for "necessary" read "unnecessary".

It is particularly fortunate that we humans live in a three-dimensional space. Huygens' principle, on the effect of sound, would not hold in an "even" space.

A Problem in the Computation of State and Federal Taxes

By J. F. THOMSON
Tulane University

An interesting problem occurs in the computation of state and Federal corporate income taxes. The Federal tax is deductible from the net income before computing the state tax, and likewise the state tax is deductible from the net income before computing the Federal tax.

The question is: What is the net income subject to state tax, after the Federal tax has been deducted? Also what is the net income taxable under Federal law, after deducting the state tax? The following is the derivation of formulas for these quantities in terms of the original net incomes taxable and the rates of taxation.

Let

a = net income taxable under Federal law, before deducting state tax.

b = net income taxable under state law, before deducting Federal tax.

F = net income taxable under Federal law, after deducting state tax.

S = net income taxable under state law, after deducting Federal tax.

x = rate of Federal tax.

y = rate of state tax.

Then $$b-Fx=S$$

and $$a-Sy=F,$$

follow from the above definitions.

Solving simultaneously, we have the formulas

$$F=\frac{a-by}{1-xy}$$

$$S=\frac{b-ax}{1-xy}$$

[NOTE: These formulas could be easily extended to take care of additional taxes; for example, if there were a city tax also in the above problem, it would only be necessary to add an additional term, get three equations, and solve for the three unknowns].

To illustrate, take the following numerical example:

$a = 16{,}828.26 \qquad\qquad b = 13{,}828.26$

$x = 13\tfrac{3}{4}\% \qquad\qquad y = 4\%.$

$$F = \frac{16{,}828.26 - (13{,}828.26)(.04)}{1 - (.1375)(.04)}$$

$$= \frac{16{,}275.13}{.9945} = 16{,}365.14$$

$$S = \frac{13{,}828.26 - (16{,}828.26)(.1375)}{1 - (.1375)(.04)}$$

$$= \frac{11{,}514.37}{.9945} = 11{,}578.06$$

Check:	*Federal*		*State*
Net income before tax........	16,828.26		13,828.26
Less tax...............(State)	463.12 ←	(Federal)	→ 2,250.20
Net taxable income...........	16,365.14		11,578.06
Income tax..............	2,250.20		463.12

$(16{,}365.14)(.1375) = \leftarrow \qquad (11{,}578.06)(.04) = \leftarrow$

An accountant was solving this problem in the following rather tedious manner:

	Federal	*13¾%*	*State*	*4%*
Tentative Income tax				
Schedule..........	16,828.26	2,313.89	13,828.26	553.13
Less tax on........	−553.13	−76.06	−2,313.89	−92.56
	16,275.13	2,237.83	11,514.37	460.57
	+92.56	+12.72	+76.06	+3.04
	16,367.69	2,250.55	11,590.43	463.61
	−3.04	−.42	−12.72	−.51
	16,364.65	2,250.13	11,577.71	463.10
	+.51	+.07	+.42	+.02
	16,365.16	2,250.20	11,578.13	463.12

He wanted a formula which would give the net income taxable (16,365.14 and 11,578.06) without having to go through this series of corrections. That the formula derived in the beginning of this paper gives the desired result has been previously demonstrated.

The formula may be also derived in the following two ways, not as easy algebraically, but interesting to the mathematician. Write

$$16{,}828.26 = a$$
$$-553.13 = -by$$
$$+92.56 = axy$$
$$-3.04 = -byxy$$
$$+.51 = axyxy$$
$$-.02 = -byxyxy$$

Adding these terms and factoring, we have

$$F = (a-by) + (a-by)xy + (a-by)(xy)^2 + \cdots.$$

Since we have a geometric series with an infinite number of terms, and ratio (xy) less than unity, the formula for the sum of the terms of such a series gives at once

$$F = \frac{a-by}{1-xy} \text{ as before.}$$

S can of course be found in like manner.

Another method of deriving this formula is as follows:

1. Start out with a, the net income taxable under Federal law, before deducting state tax.
2. The Federal tax on this is ax.
3. The reduced state income taxable is then $b-ax$.
4. The state tax on this is $(b-ax)y$.
5. The reduced Federal income taxable is then $a-(b-ax)y$.
6. The Federal tax on this is $[a-(b-ax)y]x$.
7. The reduced state income taxable is then $b-[a-(b-ax)y]x$.
8. The state tax on this is $\{b-[a-(b-ax)y]x\}y$.
9. The reduced Federal income taxable is then

$$a - \{b-[a-(b-ax)y]x\}y.$$

This process could be continued, giving a series of terms corresponding to the one previously derived.

Expanding

$$a-\{b-[a-by+axy]x\}y$$

$$a-\{b-ax+bxy-axyx\}y$$

$$a-by+axy-bxyy+axyxy$$

Factoring

$$(a-by)+(a-by)xy+(a-)(xy)^2+\cdots\cdot$$

From which the formula follows as before.

A Prophet is not without honor, save in his own, etc.

The above Bible partial quotation is well illustrated by the following incident which occurred when the writer was searching for the tomb stone of Baron John Napier, Laird of Merchiston, Edinburgh, the inventor of *Logarithms*. Arriving in Edinburgh, we decided to visit the University, to inquire where Napier had been buried. But we found that the University was closed, and the large iron gates were securely locked. But to our great delight, we saw a sign over a shop where palms and ferns were sold, and the superscription read, "William Napier, Herbologist". Entering the shop we addressed the proprietor as follows: "Good morning, Sir, we are wondering if you will tell us where your great, great, great grandfather is buried". He at once asked, "Who was he?" We told him, and to our great surprise he said he had never heard of him. Now we are wondering how the students in the University of Edinburgh had passed by that shop, not more than 50 yards from the gates of the great school, carrying their log. tables under their arms, without noticing that sign, and indeed stealing it as a souvenir, to adorn one of their rooms.—James McGiffert.

There was a young man from Trinity
Who took the square root of Infinity;
But in counting the digits,
He was seized with the figits,
Dropped signs and took to Divinity.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

G. A. Miller as Mathematician and Man: Some Salient Facts

By G. WALDO DUNNINGTON
University of Illinois

On July 31st the honored and beloved Professor George A. Miller will have attained his 75th birthday. This event will bring joy and pleasure to the wide circle of personal friends, students, scientific and mathematical colleagues where he has long been an active figure. We take pleasure in presenting the accompanying article on *Mathematical Myths*—evidence that his acute mental grasp and alertness are undimmed by advancing years. (This sketch appears as a surprise to him.) We are grateful to know that he is in the full vigor of health and hope that for many years he may remain in the midst of scholarly research—which has ever been his goal. This has been expressed quite recently in his own words in an article on *Group Theory for the Million* (Scientific Monthly, December, 1937): "One of the mainsprings of mathematical progress during all the ages; *viz.*, the search for new truths and the delight in seeing things from a new and more general point of view."

His unusually rich and useful career presents many interesting aspects. Devoted friends who have visited him in his home or office will recall battles of books and scholarly ideals. They will think of his outstanding success as a teacher of mathematics, of the international reputation he has enjoyed for many years as one of the ten leading American mathematicians.

Dr. Miller also enjoys the distinguished honor of being the only man for the publication of whose *Collected Works* the University of Illinois board of trustees has ever made appropriations of money. Such editions are extremely rare in the American scientific world and this case is unusual but doubly fortunate in the fact that Professor Miller himself does the bulk of the editorial work. Volume I ap-

peared in 1935 and has received a hearty welcome as well as highly commendatory reviews throughout the world. It is estimated by the University of Illinois Press that six large volumes of about 500 pages each will be required to complete the publication of Dr. Miller's writings, but even this will not include everything he has published. A committee of three Illinois colleagues has advised him and participated in the selection of material. The first volume contains contributions up to the year 1900 together with three valuable historical papers on the development of group theory during that period, written especially for the volume by Professor Miller. These historical articles have been warmly praised by the reviewers.

Volume II of Dr. Miller's *Collected Works* is now in press and is expected to appear soon. For it he has also written an extended memoir on the primary facts in mathematical history.

The scientific writings of Professor Miller have appeared in nearly 30 French, British, American, German, Italian, Spanish, Japanese and Indian journals and run well over 800 titles, during the last 43 years. His papers developing the theory of finite groups are estimated at about 400. They began to appear in the heyday of European influence on the budding scholarship of America. These papers exhibit great critical power in testing the validity and the value of methods. Numerous books, articles and notes testify to the broad catholicity of his interests. They comprise reviews, abstracts, expository articles and memoirs on the teaching and history of mathematics, as well as on *Determinants* (1892) and group theory. It is to be hoped that the final volume of the *Works* will contain a complete bibliography of his writings. Few scholars have been so prolific and at the same time been able to maintain such high standards. His unflagging industry has become proverbial among his friends.

As a reviewer Professor Miller, has always taken the position that the truth should be told. He feels that misinformation is worse than none at all, he is thoroughgoing, uncompromising with error and intellectual dishonesty wherever found. He goes to great lengths either to verify or disprove statements, tireless in the search for reliable accurate information, yet unfailingly kind and appreciative—giving praise to real merit where this is due. When praise comes from so fearless and honest an authority one knows it is deserved and the recipient feels correspondingly encouraged. To him a review is, what it should be, an impartial discussion of (or contribution to) the subject.

In treating the history of mathematics Professor Miller has stated a number of fundamental desiderata, at various times, which might well be summarized here: (1) There should be a clear definition of the

terms involved. (2) The reader should be encouraged to draw his own conclusions from the evidences presented and should not be confronted merely with conclusions which he is expected to accept. (3) The tendency towards deviating from the ordinary language and the ordinary grammatical constructions in speaking about mathematical questions should be noted. (4) The reader must adopt the policy of not adopting statements without verification. (5) The beginner must differentiate between working hypotheses and established facts. (6) Many of the advances in the history of mathematics as well as mathematics itself must be due to the repeated correction of errors. (7) Historical statements are frequently rich in their implications so that the student should often draw a large number of conclusions from a single statement. Miller's *A Historical Introduction to Mathematical Literature* (1916) has become a classic.

Professor Miller was born of Pennsylvania German ancestry at Lynnville. He received the A. B. (1887) and the M. A. (1890) from Muhlenberg College, which also conferred the honorary doctorate of laws on him on the 50th anniversary of his bachelor's degree. His Ph.D. was from Cumberland University in 1893. He spent one year as principal of schools in Greeley, Kansas; was professor at Eureka College, 1888-93; University of Michigan, 1893-95. The years 1895-97 were spent in advanced study and research at Leipzig and Paris. He was a member of the Cornell and Stanford faculties for several years thereafter. From 1906 until his retirement in 1931 he was professor of mathematics in the University of Illinois, being now professor emeritus. On December 23, 1909, Dr. Miller was married to Miss Cassandra Boggs, of Urbana, Illinois, known to their friends as his faithful and charming companion.

A memoir from his pen in 1900 won the International Mathematics Prize of the Cracow Academy of Sciences. He taught during several summer sessions at the Universities of California and Chicago. Was elected a fellow in the American Academy of Arts and Sciences, 1919; elected to the National Academy of Sciences, 1921; the Mathematical Association of America made him vice-president in 1916, president in 1921, and honorary life member in 1936. Corresponding member, Spanish Mathematical Society, 1920; honorary member, Indian Mathematical Society, 1922; member, Deutsche Mathematiker-Vereinigung; he served as vice-president of the American Mathematical Society, 1907-08, and as chairman of the mathematics section of the A. A. A. S., 1921-22.

It has been said that Dr. Miller is the only American who has been co-editor of the *Encyclopédie des Sciences Mathématiques*. He has

also been co-editor of the American Year Book on School Science and Mathematics. He served as mathematics chairman on the Committee of 100 on Scientific Research. Several publications were *The Algebraic Equation* in *Monographs on Topics of Modern Mathematics*, edited by J. W. A. Young (1911), *Substitution and Abstract Groups*, pp. 1-192 of Miller-Blichfeldt-Dickson. *Theory and Application of Finite Groups* (1916, Stechert reprint 1938). Professor Miller's name occupies a starred position in *American Men of Science*. Other sketches are in *Poggendorff*, v. 4-5, *Who's Who in America*, and *Leaders in Education*.

It has been vouchsafed to few scholars to produce a finer series of published works. They are the permanent record of a modest, steadfast, generous American, a man of rigid honesty and unshakable character, a scholar who eschews the bypaths of personal display and takes immeasurable pains in ferreting out the truth. Such men are always rare; they give fresh inspiration to other scholars.

In conclusion, it might be appropriate to quote several of Sir William Rowan Hamilton's lines on Fourier:

Revealed appear thy silent thoughts of youth,
As if to consciousness, and all that view
Prophetic, of the heritage of truth
To thy majestic years of manhood due:
Darkness and error fleeing far away,
And the pure mind enthroned in perfect day.

"There is a prevailing impression that once a mathematical formula has been theoretically deduced, the law embodied in the formula has been sufficiently demonstrated, provided the difference between the the 'calculated' and the 'observed' results fall within the limits of experimental error. The important point, already emphasized, is quite overlooked, namely, that any discrepancy between theory and fact is masked by errors of observation. With improved instruments, and better methods of measurement, more accurate data are from time to time available. The errors of observation being thus reduced, the approximate nature of the formulae becomes more and more apparent."—By S. W. Mellor, in *Higher Mathematics for Students of Chemistry and Physics.*

Probably the simplest transcendental number is

$$2^{\sqrt{2}}.$$

Mathematical Myths

By G. A. MILLER
University of Illinois

In a review of E. T. Bell's *Men of Mathematics*, published in the *Mathematical Gazette*, volume 21, pages 311-312 (1937), the following statements appear: "There are numerous errors of detail, unimportant in a stimulant; there is a trace, it seems to me, of an error in principle; but the author succeeds in his main aim, to portray mathematicians three-dimensionally by setting them in or against their historical background, social, cultural and philosophical." The main part of this quotation for the present purpose is the remark that the errors of detail are unimportant in a stimulant. This is probably true as regards some of the readers concerned, but there are others who will be very much annoyed by errors of detail, and whose interest in the book will be greatly diminished when they become convinced that they cannot assume that the author took a reasonable amount of care to avoid misleading remarks even when they are striking.

This stimulation engendered by accuracy both as regards the main facts and also as regards details is more lasting and more helpful to the real student than that which is due to temporary excitements based on unusual forms of expression or on overdrawn pictures of important facts or unfortunate situations. In volume 21 (1929) of the *Biometrika* Karl Pearson and G. A. Simon exhibited the myth relating to the supposed poverty with which P. S. Laplace (1749-1827) had to contend in early life. The chapter on Laplace in the volume cited at the beginning of the present article is headed "From peasant to snob", but the first sentence thereof states that he "was not born a peasant nor did he die a snob". It is true that we are told that this statement is supposed to be true "within small quantities of the second order", but it is difficult to determine what this might mean, especially since on the bottom of the same page it is stated that "his parents were peasants".

This reference to P. S. Laplace may serve to illustrate the fact that the popular recent writings of E. T. Bell tend to emphasize various mathematical myths instead of to reduce their number. This applies in particular to the myth that Pythagoras discovered the existence of irrational quantities, such as the square root of two. According to J. Tropfke's *Geschichte der Elementar-Mathematik*, volume 2, page 83 (1933), the irrationality of the $\sqrt{2}$ was probably discovered

towards the close of the fifth century B. C., while Pythagoras died around the beginning of the same century. It might be said that the nature of the writings of E. T. Bell is such that one would naturally not expect accuracy as regards many questions relating to the history of mathematics, but the reader should realize that he is in danger of contributing toward the spreading of mathematical myths when he quotes from these writings without verifying the accuracy of statements contained therein.

One of the common myths of long standing which still appears in some of our textbooks is that Napier's logarithms are our modern logarithms to the base e. Even in the second edition (1934) of *Webster's New International Dictionary*, which is being widely advertised as a supreme authority, one finds under the entry "e" the statement that e is the base of the natural or Napierian system of logarithms. It is true that this statement is contradicted under the entry "logarithm" in the same work, but such contradictory statements are naturally quite confusing to the beginner. In the present case they are especially confusing since they imply that there is an accurate rational relation between the Napierian logarithm of a general given number and the natural logarithm of the same number. The statement that tables of natural logarithms were published by John Speidell, which appears under the latter of the two given entries, is also incorrect, since John Speidell did not employ transcendental processes but based his tables on the work of Napier, while e is a transcendental number.

In recent years Professor E. Bortolotti, University of Bologna, Italy, has done much towards increasing our knowledge with respect to the history of the partial solution of the cubic equation during the sixteenth century by his countrymen Ferro, Tartaglia and Cardan. The contributions of the last of these three were especially emphasized and the stain which has been attached to his name, especially in some of our American histories of mathematics, seems to have been removed forever. In a review of a recent article by E. Bortolotti, published in the well known periodical entitled *Jahrbuch über die Fortschritte der Mathematik*, for the year 1934, it is stated on page 827 that Cardan was a throughly moral man with a severe self-discipline. The remarks which he made in connection with this severe self-chastisement seem to have given rise to the widespread view that he was actually quite immoral.

He lived in a superstitious age and seems to have shared many of the superstitions of his times but there seems to be no good reason now for regarding him as an immoral man and for assuming that he

treated Tartaglia unjustly. As in the case of the Swiss mathematician P. Guldin, who has sometimes been accused of plagiarism, the latest historical studies tend to show that there is no longer any good reason for making derogatory remarks in his biographical sketches. Since Cardan died in 1576 and Guldin in 1643 it took a long time to free their reputations from unjust criticisms and special efforts should be made to spread the truth widely so as to avoid a recurrence of these criticisms. It is questionable whether there is any other well known mathematician who has been so widely assumed to have been conspicuously immoral as H. Cardan and hence reliable information to the contrary is especially welcome, since the study of mathematics is edifying.

In 1930 W. W. Struve published a translation of the Moscow Papyrus and on page 169 he stated that the ancient Egyptians knew the fact that the area of a hemisphere is equal to that of two great circles on the surface of the sphere, and that they therefore knew a theorem which has been credited to Archimedes and constitutes one of his most conspicuous mathematical contributions. This supposed discovery by the ancient Egyptians was naturally quoted by many writers, but it was later found to be due to a mistranslation on the part of W. W. Struve. A discussion of this question appears on page 135 of the history by O. Neugebauer entitled *Vorlesungen über die Geschichte der Antiken Mathematischen Wissenschaften* (1934). On page 122 of the same work attention is called to the legend that the ancient Egyptians constructed right angles by means of a rope divided by knots in the proportion of 3,4,5, and it is here noted that this myth seems to be due to a misinterpretation on the part of M. Cantor (1829-1920), who was an influential mathematical historian.

In the first edition of volume 2 of H. Weber's well known *Lehrbuch der Algebra* (1896) it is stated, on page 54, that the natural numbers constitute a group when they are combined by multiplication. During the following year the same statement appeared in the *Bulletin* of the American Mathematical Society, volume 3, page 101. Although this statement was corrected on page 4 of the second edition (1899) of the said volume by H. Weber, it has been repeated a large number of times since then and appears on page 43 of volume 1, part 2 (1932), *Enciclopedia delle Matematiche Elementari.* It is also found on page 130 of the recent peculiar book by G. Bouligand, entitled *Premières leçons sur la théorie générale des groupes*, in which a false group theory is propounded and in which we find, page 5, the striking statement that a group is associated with every mathematical proposition. These facts show that even in our times it is difficult to prevent the spread-

ing of erroneous mathematical views when they have been started by men who command respect..

Mathematical myths are so numerous in our literature that it would not be difficult to make a large collection of them. In his recent book entitled *Erreurs de Mathématiciens des origines à nos jours* (1935), M. Lecat gave a considerable number of such myths. It seemed desirable to mention a few here, which affect especially the American student, so as to serve as a warning not to accept statements without making reasonable effort to verify them. This is especially true as regards historical statements since progress in the history of mathematics has fortunately been very rapid during recent years and hence many of the older works of reference have become unreliable. In particular, our knowledge in regard to the partial solutions of quadratic equations by the ancient Babylonians has been so much extended during recent years that one cannot rely in this respect on the histories which are ten years or more old, and even some later ones do not contain the desirable modifications as may be inferred from some of the above quotations.

The history of Greek mathematics is especially rich in mathematical myths. A recent American textbook begins with the myth that when a pupil asked Euclid "What advantage shall I get by learning these things?" he called a slave and said, "Give him a sixpence, since he must needs gain by what he learns". It is probable that some students assume that such reports are myths, but there is now a serious Greek history of mathematics involving a large number of well established historical facts which are inspiring to the serious students. The question may be asked whether the needs of such students should not receive first consideration. It must be admitted that opinions differ widely as regards such matters, but one might expect that in a textbook for college students it would be desirable to label myths as such, in case they are included, even when the text is intended for those who are not especially interested in mathematics.

A writer on the history of mathematics may find it difficult to decide whether he should confine himself to well established facts or to include also legends of long standing. Our American textbooks, including those of F. Cajori and D. E. Smith, have followed the latter course, probably with a view towards interesting a larger number of readers. Even such an eminent mathematician as F. Klein (1849-1925) remarked in the introduction to his *Vorlesungen über die Entwicklung der Mathematik* (1926-27) that it was not possible to get along without a kind of *pia fraus* in view of the fact that the inherent difficulties frequently obstructed a popular presentation if they were not ignored for a time. The history of modern mathematics is very difficult.

One of the greatest myths in mathematics is that it is a divided subject and that such names as arithmetic, algebra and geometry have a definite meaning. In such a widely used work of reference as the second edition of *Webster's New International Dictionary* it is stated under the entry "algebra" that there is an essential difference between arithmetic and algebra. On the contrary, the terms arithmetic, algebra and geometry are merely convenient names of classification and in many cases it is immaterial whether a certain development is classed with one or the other of these divisions. The main thing is to make progress and not the heading under which this advance is placed. It is, however, a matter of some interest that the mathematician has found it convenient to have various files for his technical knowledge and not only one file, although he knows very well that the divisions are mythical, except that custom has established definite rules in regard to some special cases.

A myth which has been extensively advertised recently is that there are special racial aptitudes for mathematical work along particular lines. In 1936 there was started in Germany a periodical under the title *Deutsche Mathematik*, which aims to promote the German type of mathematics and to look at everything from the viewpoint of German contributions. Even in Germany there are many who are not in sympathy with this movement and who regard it as a backward step. It used to be said that the ancient Greeks were especially gifted along the line of geometry, but the later studies made it clear that this was a superficial view since the Greeks developed a geometrical algebra. The substance of the developments is more significant than the particular form in which the developments are expressed. Even in the same country there are frequently various centers in which different lines of mathematical work are especially emphasized, and it is unnecessary to employ the hypothesis of special racial aptitudes to explain the differences in the mathematical contributions of various countries during past ages.

It is possible...to combine a respect for cathedrals with a liking for making mudpies; but for normal human beings the mudpie is the necessary first step to designing a cathedral or to respecting those who do. And our science needs its fair share of the normal-minded, to spread its ideas, to write its popularizing books. It is becoming too narrowly professional. There are not too many memoirs, but too few readable books.—Frank Morley in *Pleasant Questions and Wonderful Effects*, Bull. Am. M. S., 27 (1921) pp. 309-312.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES McGIFFERT

What is Essential in Teaching Mathematics?*

By HENRY BLUMBERG
The Ohio State University

In an address such as the one I am about to make, it is a good idea, I believe, to have some of the thoughts focus around a story. For a story is often remembered when colorless, philosophical remarks and abstract analyses fade away. And by story I do not mean the kind frequently told after dinner, told for mere entertainment without scruple as to bearing. The story I have in mind is, I think, *à propos*; I have been assured, moreover, that it is true. Here it is:

The pianist-composer Anton Rubinstein once came to America on a concert tour. Renowned master that he was, people came from great distances to hear him in his first concert in New York. One such visitor was a certain middle-aged lady from a western city who happened to put up at the same hotel where Rubinstein was stopping. Some time before the concert she lay down to rest herself from her journey. But she could not rest, for across the hallway some one was practicing piano in an exasperating monotony, as might a self-absorbed beginner. And practicing by no means *pianissimo*. When she couldn't stand it any more, she complained to the clerk, who accompanied her to the room of the amateur pianist, knocked, and as no one answered, pushed the door open. To her amazement the culprit she beheld was none other than Rubinstein himself.

But, you will ask, just what was Rubinstein doing to give our middle-aged lady the impression that he was a beginner? The answer is that he was at work on the simplest sort of finger exercise. Placing his left thumb upon the C below middle C, he beat three staccato notes on B with the index finger. Then he struck B again, this time holding the key down. Next, the middle finger did its chore of three

*Address before the Greater Cleveland Mathematics Club, February 22, 1937.

staccato notes and one held-down note on A; similarly for the fourth finger and little finger, and then similarly back over the fingers in reverse order. And then the right hand went through a corresponding ceremonial. Then came again the left hand's turn, then the right's, and so on back and forth. One prominent feature of the whole rite was the long time interval between two successive notes. And precisely this exercise, I have been told, Rubinstein would customarily execute for a space of two hours before giving a public concert.

Now what does this elementary exercise offer a concert artist? To answer this question, let us consider what is essential in learning to play the piano—to play, I mean, not merely to satisfy fixed tastes—which presumably themselves need cultivation—or a circle of friends or even a large public, since numbers don't imply artistic discrimination, especially in an age when propaganda of commerce or of ignorance plays a vast role in erecting and maintaining false values in art. Learning to play, I mean, in the light of the highest ideals of beauty and expression we can imagine—these ideals themselves, of course, ever evolving to new heights. To try to learn to play with such ideals in mind, it is essential to study tone, to acquire a fine ear for discriminating tone, to experiment on the piano for the purpose of eliciting tones that are always mellower, more resonant, more beautiful, more magnanimous. And to produce such tones one must develop a discriminating ear and a most intimate sense of the feel of fingers, hands and arms at rest and in motion. When one plays a long composition, immersed as one is in an uncritical mood, one cannot well give concentrated attention to subtle nuances of tone, one has relatively slight opportunity for fundamental improvements in finger elasticity, dexterity and power. But Rubinstein's exercise offers precisely such needed opportunity. Very few of us—few concert artists, indeed—take the pains, in the course of a whole lifetime, to listen to tone quality as the Rubinstein exercise requires.

But what is the moral? There are various morals, but first of all, I will remark, the story illustrates how important it is in every undertaking to ask again and again, What is the main thing I want to do? How few of us thus ask for our major activities, for the business of life! I do not mean ask in fleeting moments, as we, indeed, all do. I mean earnestly, searchingly, as the Bible enjoins us to ask concerning the path to Truth and Holiness—ask with our whole heart and mind and spirit.

Now I don't know that I could properly expect all this saintly zeal from a teacher of mathematics, though, in some respects, the teacher's calling likens that of the priest. But it is perhaps not unfair

to expect zeal and earnest search with reference to central things in one's profession. The first question, then, is, how many of us ask again and again, devotedly, with open mind, and in the spirit of search and adventure: What is essential in teaching mathematics? *We ourselves* must ask and seek, since the answer will not be found once and for all deposited in a book or heard as a definition out of somebody's mouth. No more than we can learn what Beauty is, in its essence and universality, by reading Plato's definition of it or hearing a great artist expound it. If we want Beauty, want to make it a living principle, we must grow in it daily, fashion new conceptions of it daily and hourly. Rubinstein evidently knew this.

There is no royal road to learning, no royal road to the essence of the learning process, no royal road to the essentials in teaching mathematics. If we want to know what these essentials are, we must ourselves become seekers, students, with the ideal, childlike attitude, open-minded, eager, unprejudiced, free from intellectual or professional pride.

Now, the seeking for what constitutes these essentials will, of course, partly consist in examining what others say, analyzing their statements with reference to our own experiences and insights, and accepting what seems appropriate. And it is at this point that it is for me to offer some of my views.

What is essential in teaching mathematics?, I ask myself. And I answer, To develop the mind. Of course, of course, most everybody will admit. But these words have varying meaning for different people; and some may subscribe to the words without bothering to take implications. To begin to clarify the issue, let us take an illustration. We all know how frequently the process of substitution comes into mathematics, say in Algebra or Trigonometry or Analytic Geometry—which can be said to have substitution as its basal and central principle—or in the Calculus or the higher branches of Analysis. Less obviously but no less vitally substitution enters Pure Geometry. Now if we have regard to clarity and system and training in principles of wide application, we would study the different forms under which substitution appears in the different branches of mathematics—first, let us say in the elementary branches. And we would study with painstaking slowness, after the manner of Rubinstein's exercise—as is seemingly not done by mathematicians—just what a particular form of substitution consists of, making sure that we have caught its essence and can recognize it, if it comes our way, in the most diverse branches of mathematics. And having become thoroughly intimate with the essence of substitution in one form we would pass on to

another form and gain a similar intimacy. Then to another form, and so on. And being so equipped, we would gradually, stepwise, train our students in similar clarity, and system and comprehensiveness. If this is done, the substitution part of a new subject will be recognized as having been already disposed of, and thus many parts of mathematics now taught as if new would be responsibly recognized as old matter. After such thorough grounding in process, it would be sufficient to spend only a fraction of the customary time, for example, on differentiation and the technique of integration and various other parts of the Calculus. And if a similar grounding is effected in what might be termed a "higher" type of substitution, a corresponding economy of time can be instituted in the teaching of graduate mathematics.*

When I speak of economizing time, I assume that the students are held to account for what they are credited to have achieved. So that just as students are now held responsible for knowledge of subject matter, they would according to the method of training in mathematical process here proposed, be held responsible for "changes of form" or substitutions of specified types, and would be expected to know how to take care of such substitutions wherever encountered. Without dignified student responsibility, the mind is not trained to develop, but sinks into habits of laziness and dullness, becoming more and more the arena of uncontrolled and ill-digested thoughts of others. No amount of knowledge can repay the loss in dulling the marvelous faculties of the mind.

Similarly, the students might, for example, be trained how to look up the meaning of a mathematical concept, make up affirmative, negative and borderline illustrations, and find one or more applications of interest. Systematic training in this procedure would help to give the students an independence which courses rarely succeed in developing. And similarly, the students might be given systematic training in the technique of generalization or other fundamental mathematical techniques.

Again, if it is our central purpose to develop the mind of our students, we must teach them to talk mathematics. The development of mind means the development of conceptions, and the development of conceptions means the development of language. It is more impor-

*A more detailed exposition of the variety of types of "substitution" or "change of form" to be met with in mathematics was given in an address on the "Change of Form" before the Ohio Section of the Mathematics Association, April, 1935. This address is now being prepared for publication. The attribute "higher" with reference to substitutions or changes of form in "advanced" mathematics seems to suggest itself as appropriate, but this appropriateness can be validated at most on the ground of relative unfamiliarity and abstraction, but not on the ground of intrinsically greater difficulty.

tant for a student to be able to explain, say, the meaning of function or derivative or probability—I mean explain it as a good teacher can explain it—than to solve hundreds of problems. A good explanation means a sensitive appreciation of the intuitive idea of the concept, justification of the rigorous formulation adopted, well chosen illustrative material, and an understanding of the nature of the divergence of the sophisticated from the intuitive concept. How rare it is for us to expect all this from our students! But if we don't, how can we be said to attend to the essence in teaching? When mathematical conceptions are entertained in ordinary language, the subconscious mind, or whatever that mysterious part of us is which works out for us the problems we set for it, can work out our mathematical problems and difficulties when the conscious mind is otherwise occupied or asleep. This doesn't happen if we have merely manipulated, because such activity lacks the simplicity and intimacy which the subconscious mind seems to require.

A fundamental consideration in developing the mind, training it in discerning essence, economizing time, and rising to higher powers is to pay the utmost possible attention to the elements of the thought process. It is here that teachers of mathematics can well afford to sit at the feet of musicians and artists to learn a valuable lesson. The varied and profound experiences in music of Anton Rubinstein taught him that a fundamental way of improving himself as a pianist was to improve the element of piano technique, namely the stroke of a single finger. And to such a conclusion not only Rubinstein, but all the great pianists have come. A sense of ryhthm and a discrimination of relative beauty are not uncommon, but the ability to produce a deep, beautiful tone almost effortlessly—and such an ability is necessary for *adequate* interpretation—is barely possessed by the greatest concert artists. In this sense, it may be truly said that ninety-five per cent or more of piano-playing consists in attaining a certain suppleness and power in fingering. In other words, the essence of improving one's piano-playing consists in improving the element. The same thing, if properly interpreted, may be said of the art of dancing and of physical culture generally. And this is true to a large degree of the graphic arts, though, unlike music and dancing, they are presented not as a succession but as a juxtaposition of elements. And what is the nature of the element in the mathematical thought process? It is the momentary character of our thinking. If we are physically relaxed, without worry, cheerful and adventuresome, with faith in ourselves, awake and alert, the element of the thought process out of which our thinking is constituted is of one kind; when these qualities change—

and this may happen any minute, just as the pianist may become more or less tense as he continues to play—the element changes. The technique of thinking depends fundamentally on the kind of thought element we allow to enter our heads. And just as piano technique can be transformed by changing the finger stroke, so our thinking can be transformed by changing the thought element. And this latter change can be more readily effected than a change in piano technique—which isn't difficult either—because the mind, it seems, is more directly pliable than the body. And just as we make progress in piano technique by training our ear to become more refined in distinguishing tone quality, so we can change our thinking by becoming more sensitive and discriminating with respect to the momentary quality of the thought process. For example, by leaving off our worst and keeping to our best we'll make great progress and forthwith shall have a better best. Again, every best of another mind will give us suggestions for improving our own best.

In view of the fundamental importance of improving the element in the mathematical thought process, it must be said that rarely in books or in teaching is mathematics presented slowly enough. And I need not add, of course, that I haven't in mind a dulling slowness, but the slowness appropriate for understanding each new item of thought as motivated by the strategy of thinking. It would be difficult to find in mathematics a parallel to Rubinstein's pre-concert exercise. But it cannot be asserted that such a mathematical parallel would be relatively of less importance.

If due attention were paid to the element, high school teachers would know how to solve *all* the so-called "reading problems" in Algebra, without fortifying themselves nights in advance against possible mishaps. How can students be sure-footed if the teachers lack systematic strategy? The same can be said of identities and "indirect" problems in Trigonometry, and likewise the supposedly more difficult problems of Analytic Geometry and the Calculus can be handled by a systematic technique. And if our mathematical strategy, on the basis of an improved element, becomes more and more conscious and clarified, we can treat likewise much of "higher" mathematics and of what is customarily deemed professional research.

I should perhaps add that in pointing out the possibility of reducing to technique things prevalently regarded as "originals" or research, I am not for diminishing originality or invention or exploration. When a province of research is reduced to technique, the scene for mind development changes. It is not in accordance with a high ideal of invention to—more or less—lazily allow the thought process to

remain vague, and nourish one's pride with the satisfaction of being original in matters that heightened clarity could reduce to a systematic procedure. Of importance it is not to be merely inventive, but to heighten our degree of inventiveness, as for example, via ***non-archimedean*** progress, which is explained below. With a due appraisal of the very limited achievements of the human intellect up to the present time—limited in relation to its infinite possibilities—there need be no fear that clearing away mysteries of one kind, even mysteries of so-called invention, will leave us no deeper or higher mysteries to explore. Indeed, the chief function of reducing invention to technique, as far as possible, is to become ready to explore higher mysteries.

In the matter of distinguishing different types of mind development, the distinction between what I shall call ***archimedean*** and ***non-archimedean*** progress is of special importance. The term archimedean relates to the axiom of Archimedes, which, when applied, for example, to the real continuum, asserts that if a and b are any two positive numbers, some integral multiple of a will exceed b. Suppose functions are arranged according to their eventual magnitude at infinity, that is to say, the function $f(x)$ will be said to exceed the function $g(x)$ if the number $f(x)$ exceeds the number $g(x)$ for all sufficiently large x. With such an understanding, the system of functions $x, x^2, \cdots x^n \cdots$ is a non-archimedean system of magnitudes inasmuch as the function $x!$ exceeds every integral multiple of x^l if $k > l$. Another system of magnitudes that may be interpreted to be non-archimedean is the set of openings between a fixed straight line l and all other straight lines in the plane together with the openings between l and all circles tangent to it.

Now when I speak of archimedean progress, I mean the type of progress symbolized, say, in the passage from three to four, or to a thousand, or three to 10^{106} for that matter. Such progress, no matter how impressive in one respect, is in another respect nothing to be proud of. Non-archimedean progress would be symbolized, say, in the passage from 3 to x or from x to x^2—in accordance with our previous definition of the relative eventual magnitude of two functions at infinity. In piano playing, we have an example of archimedean progress, when, after one has learned to play say twenty pieces, one learns a new piece for ten new pieces or, for that matter, a thousand, assuming that the manner of rendering—tone, finger dexterity and interpretative insight—are only moderately changed. An example of non-archimedean progress is the acquisition of a new kind of tone, say one as much deeper and mellower than that of the average pianist as the voice of a fine singer is in comparison with an untrained ordinary voice. Again, we have

a case of non-archimedean progress when a pianist discovers a strikingly shorter path to increased suppleness and power of muscles, thus securing the means for accelerated changes in technique. In mathematics, we have a case of archimedean progress when one digests another page, say, of the Calculus or ten such pages or five hundred for that matter; or when one takes successfully—in the customary sense—a new course or a dozen new courses. On the other hand, we have a case of non-archimedean progress when there is a change from an ordinary interest in mathematics—even the interest of the average teacher of college mathematics—to the interest of the true amateur, the kind of interest one has in an avocation one is very fond of, an interest free of motives of vain display, or financial gain or professional advancement. Another example is in the passage from the capacity of understanding the argument in a mathematical text-book or treatise to that of devising the proofs for one's self, perhaps with the aid of occasional hints. Or acquiring an altogether new depth of concentration in one's work, every word and idea now making an incomparably greater and more distinct impression than before. We have perhaps all experienced certain moods of deep repose and high alertness, when familiar sensations take on a new distinctness, and numerous impressions, ordinarily unnoticed, reach the mind—moods when landscapes offer new experiences of pattern and color, and the tones of an orchestra seem to vibrate with the intimacy of one's innermost thoughts. A change from ordinary states of consciousness to such lucid states may be regarded as non-archimedean. It would be going after non-archimedean progress to be in quest of such lucid states, to seek their genesis, the technique of inducing them, enhancing their clarity and making them more enduring. How different such a quest from the customary quest of additional knowledge!

It may be noted that the terms archimedean, and non-archimedean are to be taken in a relative sense. It is a question here, of course, not of merely accurate, but of useful discriminations.

If the essence of living is taken to be mind development, some people may, in a certain sense, be said to live more in one day than some others in ten years—because, though the others may progress continually, they may be going the archimedean way, unaware of the rate of progress and depth of comprehension open to those who go the non-archimedean way.

Since it is of preeminent importance in the learning process to cultivate non-archimedean progress, you may ask, how can such progress be attained? This is a large question, and in answer I will give just a number of indications. First, it means much already to

decide quietly and earnestly to go the non-archimedean way. "Seek and you shall find" has in it deep truth and applies in the present instance. We can be assured that we shall find if we clarify our will and invest our intelligence. But though non-archimedean progress is a priceless jewel, it is not on that account difficult of attainment. Here again the Rubinstein story offers a significant lesson. Does it seem so hard to do what he did—or something similar, something more flexibly experimental with reference to the element of tone production? No, but one must first have acquired deep faith in building piano technique *via* experimentation with the element. Besides, devotion, patience and high ideals are necessary. Indeed, if anything is hard and fruitless it is to work without sound fundamental principles. And if we mean to develop the mind, it is working without fundamental principles to be unaware of the non-archimedean path. We need, of course, to abandon fear that we cannot attain. I prefer to assume that the Lord harbors no prejudices against us, that He does not cast us out if we don't cast ourselves out. If we seek earnestly we shall have daily new intimations and insights. We shall learn to appreciate more and more what care of the body means for the growth of the mind—I mean not athleticism, but the kind of attention that makes mind development central, that produces a mind delighting more and more in adventures in consciousness. We shall learn daily new ways of controlling our thoughts, and enhancing our creativeness for example, by eliminating negative thoughts except when seemingly indispensable. We shall learn more and more the fundamental importance of high attainment in the elementary step that is basic to a particular activity, the step which if mastered immeasurably economizes our energies and lifts our plane of attainment. We shall find daily new ways to that state of great peace and high alertness in which new ideas come readily from within ourselves. And we shall find daily new ways of surrounding ourselves with a more congenial and inspiring atmosphere, physical, intellectual and spiritual.

In conclusion, a few words on the relation of mathematics to the social-economic process, to our culture. If mathematics is to exert its due influence upon our civilization, the teachers of mathematics need to realize, ever more clearly, the peculiar and unique significance of mathematics as a means for cultivating the mind. No other subject offers a like spectacle of conceptual architecture, symphonic in elaborateness and subtlety of patterning. Because mathematics develops concepts in a way peculiar to itself, it has great lessons to teach the race on how to think,—on a plane quite different from that of certain popular books on this subject—on the nature of intellectual intuition

and invention, on going from old to new things *via* a conscious and developing technique, and on adventures in the new by learning better how to court the mathematical Muses. These and other great opportunities mathematics offers for the development of the mind—*offers* but cannot guarantee. All depends on how mathematics is learned or taught. A technician, no matter how brilliant, has no credentials for true insights in fields outside of his specialty. Unless the elements of the mathematical thought process are well understood and generalized we cannot expect trustworthy transfer of such functioning of the mind to other fields. But when the opportunities which mathematics offers are properly understood, mathematics can begin to exert its merited influence upon our culture. No doubt tremendous forces operate against our attempting to make such ideals potent in our lives. But for each individual the salient question is whether he abets these dark, enslaving forces or does his bit to combat them. We may be inclined to attempt to change the ways of others, or to use the methods of the politician in trying to promote our professional vested interests; but it is truly more practical and profitable to change ourselves. It may be well to search our hearts and inquire whether we ourselves properly appreciate mathematics. After all, no influence is comparable to the real character, the real life of the teacher.

Students in college are often surprised, and always delighted, to learn that there are many very simple short cuts to abbreviate arithmetical calculations. Teachers of the junior and senior high school classes would gain much admiration and power, if they would teach their pupils some of these shortcuts. A very simple one relates to squaring numbers in the fifties.

To square a number in the fifties, simply add the unit figure to 25, and annex the square of the unit figure. Thus to square 56, we merely add 6 to 25, giving us 31, and annex the square of 6, which is 36, and thus we obtain

$$56^2 = 3136.$$

If the square of the unit figure is less than 10, we must insert a cipher before the square of the unit figure. Thus to square 53, we add 3 to 25, giving 28, and annex a cipher and the square of 3, which is 9. Thus we get

$$53^2 = 2809.$$

—James McGiffert.

Mathematical World News

Edited by
L. J. ADAMS

Professor C. D. Smith of Mississippi State College attended the Mississippi Educational Association, April 17, to take part in a conference on Mathematics in the New Program. Preliminary recommendations of the Joint Commission were discussed with a view to seeking cooperation of the State Department of Education regarding allocation, content, and required courses with designated electives in the high school grades. A committee was appointed to report next year on the work of the Joint Commission.

Professor Marcus Skarstedt of Whittier College in Whittier, California offers a mathematics course of the new type. The methods of the course include the use of lectures, assigned reading, informal discussions, demonstrations of various kinds and notebooks. The titles of the main subdivisions are: Nature, Use and Difficulty of Mathematics; Notation and Symbolism of Mathematics; Customary Divisions of Mathematics; Arithmetic; Algebra; Geometry, Trigonometry and Vectors; The Calculus and Higher Analysis; Mathematics and the Job; Mathematics and other great fields of knowledge; Mathematics and life.

The collected reports of the Second International Congress of Mathematical Recreations (Paris, 1937) are offered for sale by Sphinx, 173 Avenue Longchamps, Bruxelles. It is a volume of 104 pages. They also have the collected reports of the first Congress (Bruxelles, 1935).

The quarterly journal *Scripta Mathematica* is planning the publication of a portfolio of photographs and short biographical sketches of eminent physicists.

Dr. Daniel Buchanan, dean of the faculty of arts and sciences at the University of British Columbia, will conduct two courses in astronomy during the coming summer session of the University of California at Los Angeles.

Professor L. N. G. Filon, president of the Mathematical Association of England, died December 29, 1937, only a week before he was

scheduled to give his presidential address at the annual meeting of the Association.

The annual meeting of the Mathematical Association of England was held on January 4-5, 1938. Among the papers presented were:

1. *Mass and Weight in Newtonian Mechanics.* Professor L. N. G. Filon (read by Professor G. B. Jeffery).
2. *The Mechanical Integration of Differential Equations.* Professor D. R. Hartree.
3. *The Relative Value of Pure and Applied Mathematics.* Discussion directed by Professor E. H. Neville.
4. *The Law of Quadratic Reciprocity.* Mr. N. R. C. Dockeray.
5. *The Feuerbach Quadrilateral.* Mr. N. M. Gibbins.
6. *The Relevance of Mathematical Philosophy to Mathematical Teaching.* Mr. M. Black.

The semi-centennial celebration of the founding of the American Mathematical Society will be held in New York City on September 6-9, 1938. Professor R. C. Archibald will describe the history of the Society, Dean G. D. Birkhoff will review the progress of mathematics during the last fifty years, and work in special fields will be related by Professors E. T. Bell, G. C. Evans, E. J. McShane, J. F. Ritt, J. L. Synge, T. Y. Thomas, Norbert Wiener and R. L. Wilder.

The M. Bocher Prize is to be awarded at the 1938 annual meeting of the American Mathematical Society to the author of a paper on Analysis during the period 1933-1937.

The Southeastern Section of The Mathematical Association of America and the Georgia Academy of Science held meetings at the Georgia School of Technology, Atlanta, Georgia, April 1-2. More than 200 mathematicians from seven southern states and 100 scientists from Georgia were present. Fifty-three papers were presented. Professor F. D. Murnaghan of Johns Hopkins University was the joint visiting speaker. His addresses were *The basic ideas of Arithmetic and Algebra* and *Finite Deformations of an Elastic Solid.*

Professor E. L. Rees, who has been connected with the Mathematics Department of the University of Kentucky since 1907, has resigned. He expects to devote his time to business and travel. He is now traveling in South America.

Dr. C. G. Latimer is Visiting Lecturer in Mathematics at the University of Wisconsin for the second semester, 1937-1938.

The Department of Mathematics, University of Kentucky, has granted graduate assistantships to Mr. Laverne Tripp and Mr. W. H. Clatworthy for the second semester, 1937-1938.

The Mathematics Department, University of Kentucky, offers the following courses in advanced mathematics during the summer session, 1938:

First Term—(June 13-July 16)
Projective Geometry by Dr. P. P. Boyd.
Theory of Equations by Dr. H. H. Downing.
Calculus of Variations by Dr. H. H. Downing.

Second Term—(July 18-August 20)
Infinite Series by Dr. F. E. LeStourgeon.
Higher Algebra by Dr. C. G. Latimer.

The Minister of the Interior in Germany has ordered that the schools replace the 360 degree circle by a 400 degree circle. It is intended that several years elapse before the change will be made obligatory for German navigators. This change has been proposed in other countries at various times and there are many who believe that the decimal system would be preferable to the sexagesimal scheme now used.

The engineering and mathematics section of the Southern California Junior College Conference met at Pasadena on April 23, 1938. There were two addresses:

1. *Factoring Large Numbers.* Professor Paul H. Daus, University of California.
2. *Fermat's Last Theorem, and the Origin of the Theory of Algebraic Numbers.* Professor Morgan Ward, California Institute of Technology.

The American Mathematical Society met at the University of California (Berkeley) on April 9, 1938. Some twenty-five research papers were presented. Professor A. D. Michal, California Institute of Technology, delivered an hour address on *General Differential Geometries* and related topics.

The American Mathematical Society met at the University of Chicago on April 8-9, 1938. In addition to the usual presentation of research papers Professor C. G. Latimer, University of Kentucky, conducted a symposium on the *Arithmetic of Generalized Quarternions.*

The annual meeting of the Indiana Section of the Mathematical Association of America was held on May 6-7, 1938 at the Indiana State Teacher's College in Terre Haute. Addresses scheduled included:

1. *Report on the work of the Joint Commission on the Place of Mathematics in the Secondary Schools.* Professor K. P. Williams, Indiana University.
2. *Report on the study of a committee of the American Association of Teachers Colleges on Desirable Attainments for Teachers of Secondary Mathematics.* Professor L. H. Whitcraft, Ball State Teachers College.
3. *The Three Classical Problems of Geometry.* Professor Karl Menger, Notre Dame University.

In addition, seven mathematical papers were presented.

The Rocky Mountain Section of the Mathematical Association of America met at the University of Colorado on April 15-16, 1938. The program included joint sessions with the Colorado School and College Conference and with the mathematics section of the eastern division of the Colorado Educational Association. Among the addresses presented were the following:

1. *The Type B Gram-Charlier Series.* Professor Leo A. Aroian, Colorado State College.
2. *Teaching Large Sections in Freshman Mathematics.* Professor O. H. Rechard, University of Wyoming.
3. *Mathematics and Science After Keyser.* Professor S. L. Macdonald, Colorado State College.
4. *Roots of Agebraic Equations with Complex Coefficients.* Professor A. J. Kempner, University of Colorado.
5. *Slide Rules for the Solution of Two Problems in Spherical Trigonometry.* Professor Ivan L. Hebel, Colorado School of Mines.
6. *Lagrangean Multipliers—an Exposition.* Professor C. A. Hutchinson, University of Colorado.
7. *Cooperation Between High School and College Teachers of Mathematics.* A. E. Mallory, Colorado State College of Education, and F. A. St. John, South High School, Denver.
8. *The Facts of Geometry Most Needed by the Students of College Mathematics.* Professor A. J. Lewis, University of Denver.

9. *Preparation of Teachers of Mathematics with Reference to the Requirments of the North Central Association.* A. E. Mallory, Colorado State College of Education.
10. *Approximate Numbers.* J. R. Britton, University of Colorado.
11. *Report on Work of National Joint Commission.* C. A. Hutchinson. University of Colorado.

Massachusetts Institute of Technology

First period, June 13-July 23:

In addition to courses in calculus and differential equations covering the work of the first two years, the following advanced courses are offered: By Professor Franklin, Advanced calculus, Mathematical reading, and Differential equations for electrical engineers.

Second period, July 25-Sept. 3:

Courses covering the first two years repeated, (except first semester calculus). The following advanced courses will be given: By Professor Struik, Advanced calculus; by Professor Zeldin, Vector analysis.

August 1-Sept. 3:

Courses in algebra, solid geometry and trigonometry, in preparation for fall extrance examinations in these subjects.

How many high school students know that to square any number ending in $\frac{1}{2}$, we merely multiply the integral portion of the number by the next higher integer, and add one fourth. For example $7\frac{1}{2} \times 7\frac{1}{2} = 56\frac{1}{4}$.

This is worth while, for the necessity of using the formula for the area of a circle, πr^2, is so frequent. If the diameter is an odd integer, the radius ends in $\frac{1}{2}$. A slight extension gives us the square of any number ending in 5. Thus to square 75, we simply multiply 7 by 8, getting 56, and annexing 25, so that we get $75^2 = 5625$.

Leibnitz produced one of the most remarkable things in all of mathematics: the extremely simple series for π:

$$\frac{\pi}{4} = 1 - 1/3 + 1/5 - 1/7 + \cdots$$

This was also given by Gregory a few years before Leibnitz published it.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

SOLUTIONS

No. 48. Proposed by *William E. Byrne*, Virginia Military Institute.

Given the function $f(x)=(1+x)^{1/x}$, $x=0, f(0)=e$. Find the Taylor expansion of $f(x)$ valid for x near 0.

Solution by the *Proposer*.

If $\qquad f(x)=(1+x)^{1/x},\ x\neq 0, f(0)=e,$

then $\qquad \log f(x)=\dfrac{1}{x}\log(1+x),\quad x\neq 0$

$$=\frac{1}{x}\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^n\frac{x^{n+1}}{n+1}+\cdots\right]$$

$$=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots+(-1)^n\frac{x^n}{n+1}+\cdots$$

The series development for $\log f(x)$ holds even when $x=0$.

Let $\qquad \log f(x)=1+u$

then $\qquad f(x)=e^{1+u}=e\left[1+u+\dfrac{u^2}{2!}+\cdots+\dfrac{u^n}{n!}+\cdots\right]$

$$=e\left[1-\frac{x}{2}+\frac{11}{24}x^2-\frac{7}{16}x^3+\cdots\right]$$

A limited expansion is useful in studying the convergence or divergence of a series whose general term is given by

$$u_n=\left(1+\frac{1}{n}\right)^n-e$$

or

$$u_n=\left(1+\frac{1}{n}\right)^n-e+\frac{a}{n}$$

(find a so that the series will be convergent).

No. 182. Proposed by *A. Moessner*, Nurnberg-N, Germany.

What is the solution of the identities:

$$A_1\cdot A_2=A_1+A_2$$

$$A_1\cdot A_2\cdot A_3=A_1+A_2+A_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_1\cdot A_2\cdot A_3\cdots A_x=A_1+A_2+A_3+\cdots+A_x\ ?$$

Partial Solution by *E. P. Starke*, Rutgers University.

Evidently

(1) $$1/A_2=1-1/A_1.$$

If $r>1$, we may put

(2) $$\prod_1^r A_n=\sum_1^r A_n=A_r+\sum_1^{r-1} A_n=A_r+\prod_1^{r-1} A_n.$$

If we multiply through by A_r, there results

(3) $$A_r\cdot\prod_1^r A_n=A_r^2+\prod_1^r A_n.$$

Analogous to (2), we have $\prod_1^{r+1} A_n=A_{r+1}+\prod_1^r A_n$, or

(4) $$A_{r+1}\cdot\prod_1^r A_n=A_{r+1}+\prod_1^r A_n.$$

The elimination of ΠA_n from (3) and (4) produces

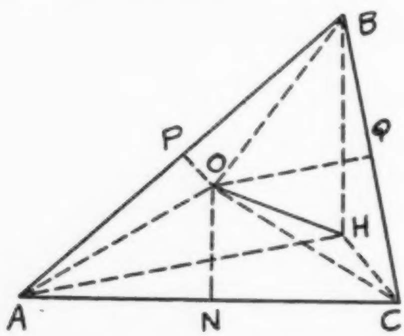
$$\frac{A_r^2}{A_r-1}=\frac{A_{r+1}}{A_{r+1}-1}\quad\text{or}\quad\frac{1}{A_{r+1}}=\frac{1}{A_r^2}-\frac{1}{A_r}+1,\quad r>1,$$

which with (1) is a reduction formula by which all values of A_r may be obtained in terms of an arbitrary A_1. Explicit formulae for the A_r in terms of A_1 are not given.

No. 199. Proposed by *Walter B. Clarke*, San Jose, California.

Given the triangle ABC with $a<b<c$, and $B=60°$. Show that: (1) Side a plus intercepted length of the Euler line equals side c. (2) The sum of the three line segments from the orthocenter to the vertices is to the sum of the sides as the circumradius is to side b.

Solution by *C. W. Trigg*, Los Angeles Junior College.



Since $B=60°$, $R=b/2\sin B=b/\sqrt{3}$, and $b^2=a^2+c^2-2ac\cos B$ $=a^2+c^2-ac$. Let Q, N and P be the midpoints of BC, CA and AB, respectively. Let O be the circumcenter and H be the orthocenter.

(1) From pages 175 and 165 of Johnson's *Modern Geometry*, M being the median point,

$$\overline{OH}^2=9\,\overline{OM}^2=9R^2-(a^2+b^2+c^2).$$

When the established values are substituted in this relationship,

$$\overline{OH}^2=2b^2-a^2-c^2=a^2-2ac+c^2, \text{ so } OH=c-a \text{ and } a+OH=c.$$

(2) $\angle AON=\angle B=\angle NOC=60°$, so $\angle OAN=\angle OCN=30°$. Hence $ON=R\sin 30°$, $OQ=R\sin(C-30°)$, and $OP=R\sin(A-30°)$ or $R\cdot\sin(30°-A)$, where the latter value is unnecessary if OP is considered as a directed segment. Upon substitution in the well-known relationship, $AH+BH+CH=2(OQ+ON+OP)$

$$=2R[\sin(C-30°)+\sin 30°+\sin(A-30°)]$$

$$=2R[\sin 30°+\sin A\cos 30°-\cos A\sin 30° \\ +\sin C\cos 30°-\cos C\sin 30°]$$

$$=2R\left[\frac{b}{2b}+\frac{a\sin 60°\cos 30°}{b}+\frac{c\sin 60°\cos 30°}{b} \\ -\sin 30°(\cos A+\cos C)\right]$$

$$=\frac{2R}{b}\left[\frac{b}{2}+\frac{3a}{4}+\frac{3c}{4}-\frac{b}{2}\left(\frac{b^2+c^2-a^2}{2bc}+\frac{a^2+b^2-c^2}{2ab}\right)\right]$$

$$=\frac{R}{b}[a+b+c]=2Rs/b.$$

The final reduction is accomplished by use of the relation $b^2=a^2+c^2-ac$.

No. 200. Proposed by *Robert C. Yates*, University of Maryland.

Consider the directed triangle ABC.

(1) Give the construction of the circle through A and B which makes equal angles with AC and CB.

(2) Show that the three circles so constructed meet in a point.

Solution by *Walter B. Clarke*, San Jose, California.

(1) Let the perpendicular bisector of AB meet the circumcircle of ABC at P. This point P is the center of the required circle. For, if lines be drawn at A and B perpendicular to AP and BP respectively, meeting at the point H on the circumcircle of ABC, then it is apparent that $\angle AHB=C$ and $\angle HAB=\angle HBA=(\pi-C)/2$. Thus we have: (assuming $A<B$)

$$\angle HAC=\angle HAB-A=(\pi-C)/2-A=(B-A)/2,$$

and $$\angle CBH=B-\angle HBA=B-(\pi-C)/2=(B-A)/2.$$

(2) It is known* that the circle with center P and radius PA passes through the incenter of ABC. Accordingly, since (1) was obtained for an arbitrary side, all three such circles meet at the incenter.

Also solved by the *Proposer*.

*See Altshiller-Court, *College Geometry*, Art. 102.

No. 203. Proposed by *V. Thébault*, Le Mans, France.

Spread the first 18 integers upon the perimeter of a triangle in such a fashion that

$$a+b+c+d+e+f+g$$
$$=g+h+i+j+k+l+m$$
$$=m+n+p+q+r+s+a$$

$$a^2+b^2+c^2+d^2+e^2+f^2+g^2$$
$$=g^2+h^2+i^2+j^2+k^2+l^2+m^2$$
$$=m^2+n^2+p^2+q^2+r^2+s^2+a^2.$$

Solution by the *Proposer*.

Upon adding, member by member, the equations

$$a+b+c+d+e+f+g=A \qquad a^2+b^2+c^2+d^2+e^2+f^2+g^2=B$$

$$g+h+i+j+k+l+m=A \qquad g^2+h^2+i^2+j^2+k^2+l^2+m^2=B$$

$$m+n+p+q+r+s+a=A \qquad m^2+n^2+p^2+q^2+r^2+s^2+a^2=B$$

we have

$$(\textstyle\sum a)+q+m+a=3A, \qquad (\textstyle\sum a^2)+q^2+m^2+a^2=3B,$$

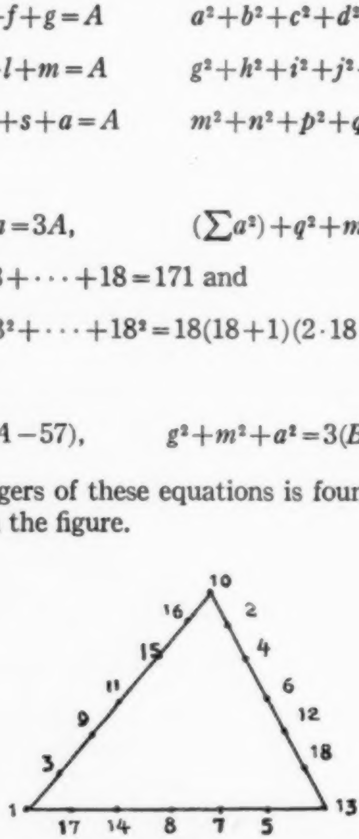
where $\sum a=1+2+3+\cdots+18=171$ and

$$\textstyle\sum a^2=1^2+2^2+3^2+\cdots+18^2=18(18+1)(2\cdot 18+1)/6=2109.$$

Thus we obtain

$$g+m+a=3(A-57), \qquad g^2+m^2+a^2=3(B-703).$$

The solution in integers of these equations is found without difficulty to be as indicated in the figure.



No. 207. Proposed by *W. V. Parker*, Louisiana State University.

Show that $y=k\cdot x\cdot \log x$ has points of maximum and minimum curvature if $k>2/3$. Determine all positive rational values of k such that $\log x$ is rational at these points.

Solution by *W. B. Brown*, Mississippi Woman's College.

The curvature K is easily found to be given by:

$$K/k=(1/x)\cdot[1+k^2(\log x+1)^2]^{-3/2}. \tag{1}$$

To determine the points where K is a maximum and minimum, the derivative of (1) is equated to zero. That is,

$$0=(1/x^2)[1+k^2(\log x+1)^2]^{-3/2}\cdot\left[1+3\frac{k^2(\log x+1)}{1+k^2(\log x+1)^2}\right] \tag{2}$$

The first factor vanishes only when $x=\infty$; the second only when $x=0,\ \infty$; the third therefore is the one of interest. This last factor vanishes whenever

$$k^2\log^2 x+5k^2\log x+(4k^2+1)=0. \tag{3}$$

For $\log x$ to be rational the discriminant of (3), $\Delta=k^2(9k^2-4)$, must be a perfect square. That is, $9k^2-4=n^2/m^2$, where n and m are integers. Hence

$$k=(1/3m)\sqrt{(4m^2+n^2)}. \tag{4}$$

Obviously, for the discriminant of (3) to be positive, $k>2/3$. When $k=2/3$, $\Delta=0$, and $\log x=-5/2$ so that this value of k would appear to satisfy the stated conditions.*

In order for other values of k to be positive and rational $4m^2+n^2$, (an integer) must be a perfect square. The numbers that satisfy these conditions are given by the following sequences:

n	3	5	7	.	.	.
m	2	6	12	.	.	.
$\sqrt{(4m^2+n^2)}$	5	13	25	.	.	.
$3m$	6	18	36	.	.	.
k	5/6	13/18	25/36	.	.	.

*This value of k, however, does not produce a change of sign in the derivative of the curvature.—Ed.

Thus $$k=(1+2p+2p^2)/3p(1+p), \quad (p=1,2,3,\cdots).$$

Also solved by the *Proposer* and *H. T. R. Aude*, who makes the following remarks:

Negative values of k give curves which are respectively symmetrical with respect to the X-axis with those curves which have the same numerically positive values of k. The real values of maximum and minimum radii of curvature, therefore, exist only when $k>|2/3|$.

Rational values for log x exist if and only if $9k^2-4$ is a perfect square. That is, whenever there exists a rational number h such that $9k^2=h^2+4$. This means that the number triad $(3k, h, 2)$ is such that they can form a rational right triangle. It is well known that the necessary and sufficient condition for this is given by the relation

$$3k : h : 2=(r^2+s^2) : (r^2-s^2) : 2rs$$

where r and s are positive relatively prime integers. If the further restriction is added, that r and s are not both odd, then there is no repetition of such triads. Thus

$$k=(r^2+s^2)/3rs$$

represents the required values.

No. 209. Proposed by *C. N. Mills*, Illinois State Normal University.

The axes of three right circular cylinders, with equal radii R, are concurrent and mutually perpendicular. Show that the volume of the solid bounded by the cylinders is $8R^3(2-\sqrt{2})$.

Solution by *H. T. R. Aude*, Colgate University.

Let the axes of the three cylinders be the rectangular coordinate axes, then the volume in the first octant is symmetrical with respect to the plane $x-y=0$. The boundaries of one of these parts, using cylindrical coordinates (r, Θ, z) are the plane $z=0$ and the cylindrical surface $z=(R^2-X^2)^{\frac{1}{2}}=(R^2-r^2\cos^2\Theta)^{\frac{1}{2}}$, below and above, respectively; the two planes $\Theta=0$ and $\Theta=\pi/4$, which meet in the z-axis; and the cylindrical surface $(x^2+y^2)^{\frac{1}{2}}=r=R$.

One sixteenth of the desired volume is given by the double integral

$$\frac{1}{16}V=\int_0^{\pi/4}\int_0^R (R^2-r^2\cos^2\Theta)^{\frac{1}{2}}\, r\,dr\,d\,\Theta.$$

Whence the entire volume is

$$V = -\frac{16}{3} R^2 \int_0^{\pi/4} \sec^2\theta(\sin^3\theta - 1)d\theta = 8R^3(2-\sqrt{2}).$$

Also solved by *W. B. Brown*, *Albert Farnell*, and *Davis P. Richardson.*

No. 213. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Prove that the product of two numbers, each of which is of the form p^2+pq+q^2, is of this form in four ways.

Solution by the *Proposer.*

We have at once

$$(a^2+ab+b^2)(c^2+cd+d^2) = x^2+xy+y^2,$$

where
$$\begin{aligned} x,\ y = (ac+ad+bc), &\quad (bd-ac);\\ (bc+bd+ac), &\quad (ad-bc);\\ (bd+bc+ad), &\quad (ac-bd);\\ (ad+ac+bd), &\quad (bc-ad). \end{aligned}$$

Editor's Note: The four representations can be obtained from any one of them by the interchange of a and b, of c and d, or of a, b and c, d. A fairly extended treatment of numbers, similar to the above, which form a domain with respect to multiplication is given in Carmichael, *Diophantine Analysis*, (1915), Chapter II.

No. 216. Proposed by *V. Thébault*, Le Mans, France.

What are the last three digits of the number 7^{9999}?

Solution by *D. P. Richardson*, University of Arkansas.

Since $7^4 = 2401$, the last two digits of $(7^4)^n$ will be 01, while the preceding digit will be the last digit of the product of n by 24. Therefore the last three digits of $7^{10,000} = (7^4)^{2500}$ are 001. Now dividing $7^{10,000} = 10^3k+1 = 1001+10^3(k-1)$ by 7, we have $7^{9999} = 143+10^3(k-1)/7$. Thus the required last digits are 143, as $(k-1)/7$ is evidently an integer.

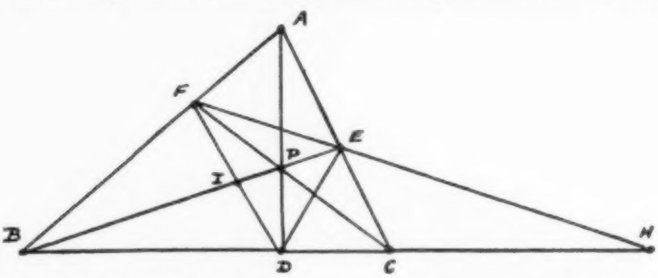
Also solved by *H. T. R. Aude*, *G. W. Wishard*, *E. C. Kennedy*, *C. W. Trigg*, and the *Proposer.*

No. 214. Proposed by *David Amidon*, Central High School, Newark, New Jersey.

On the altitude AD of triangle ABC, select an arbitrary point P. Let BP meet AC in E and CP meet AB in F. Show that (a) angle EDF is bisected by AD and (b) EF and BC meet in a fixed point however P may be chosen.

Solution by *Henry Schroeder*, Louisiana Tech., Ruston, La.

The line that joins the intersections of the opposite sides of a quadrilateral is divided harmonically by the diagonals produced.* Thus $(BCDH)$ is a harmonic range and FB, FD, FC, FH is a harmonic pencil. Since any line cutting a harmonic pencil is divided harmoni-



cally by it, then $(BPIE)$ is also a harmonic range. Moreover, if two conjugate rays of a harmonic pencil are perpendicular they bisect the angles formed by the other two rays. Accordingly, DP bisects angle IDE.

As P moves up and down AD, the points B, C, and D remain fixed. Thus the fourth point H of the harmonic range is fixed.

Also solved by *Walter B. Clarke, Albert Farnell, Yudell Luke,* and *W. T. Short.*

Late Solution: No. 146 by *Yudell Luke.*

PROPOSALS

Note: No solutions have appeared for the following numbered problems: 2, 7, 15, 20, 30, 31, 33, 38, 39, 40, 42, 57, 58, 59, 60, 62, 70, 83, 86, 97, 98, 105, 107, 112, 114, 132, 133, 134, 150, 155, 162, 177, 178, 180, 187, 192, 193.

Readers are urged to give these problems consideration and submit solutions or discussions. Unsolved problems from the early issues of this Magazine will be reprinted from time to time as space permits.

*See Altshiller-Court, *College Geometry*, Art. 262, and Art. 263, p. 140.

No. 2. Proposed by *T. A. Bickerstaff*, University of Mississippi.

A fox starts swimming at a given linear velocity from the center of a circular pool of given radius, keeping his direction always towards a goose which is swimming with the same linear velocity around the edge of the pool. Will the fox overtake the goose? If so, when? Discuss or find the equation of his path.

[Editor's Note: This is the familiar ***Problem of Pursuit*** that has had a long and honorable history. It has appeared frequently in one form or another in many periodicals and books on recreations and is credited by some writers to Leonardo de Vinci.* See Cohen's ***Differential Equations***, p. 173. F. V. Morley has given in American Mathematical Monthly, 1921, pp. 54-61, a beautiful graphical discussion of this case. See, also, in the same volume, pp. 91-93, interesting historical notes by Archibald. An account of the undeveloped technical and historical features would be particularly welcome to this Department.]

No. 7. Proposed by *T. A. Bickerstaff*, University of Mississippi.

A bullet strikes the face of a wooden beam with a velocity of 1000 feet per second. It is brought to rest at the opposite face after passing through 6 inches of wood in 1/24 second. Find its velocity at the instant when it had passed through 1 inch of wood if the retardation is constant.

No. 236. Proposed by *V. Thébault*, Le Mans, France.

Form a perfect square of nine digits for each of the forms

$$aabbccdd5 \quad \text{and} \quad 5qqrrsstt.$$

No. 237. Proposed by *E. P. Starke*, Rutgers University.

Show that no integer $x^2+y^2+z^2+1$ can be a multiple of 8.

No. 238. Proposed by *V. Thébault*, Le Mans, France.

Being given a triangle ABC, consider the following ellipses: focii B and C and passing through A; focii C and A and passing through B; focii A and B passing through C.

(1) The product of the minor axes of these ellipses is equal to the product of the diameters of the escribed circles of the triangle.
(2) The orthogonal projections of the radii of curvature of these ellipses at A, B, C upon the adjacent sides are harmonic means between the two sides.
(3) Determine the points of intersection of the ellipses.
(4) Examine the case where the ellipses are replaced by hyperbolas.

*See e. g. Mathematical Gazette, (1930-1), p. 436.

No. 239. Proposed by *V. Thébault*, Le Mans, France.

What is the largest term in the expansion of $(3+4)^{34}$?

No. 240. Proposed by *G. W. Wishard*, Norwood, Ohio.

If A_1, B_1 and C_1 are 1, 5 and 7, respectively, A_1^2, B_1^2 and C_1^2 are in arithmetical progression with common difference $24=24\cdot 1$. If $A_2=C_1$, $B_2=B_1+8$, $C_2=C_1+10$, then A_2^2, B_2^2 and C_2^2 have a common difference $120=24(1^2+2^2)$. Similarly if $A_3=C_2$, $B_3=B_2+12$, $C_3=C_2+14$, then A_3^2, B_3^2 and C_3^2 have a common difference

$$336=24(1^2+2^2+3^2).$$

Prove that this sequence extends to infinity and that A_n, B_n and C_n are relatively prime.

No. 241. Proposed by *W. V. Parker*, Louisiana State University.

In a circle, a diameter $AB=d$ is extended one unit to C. The tangent at A and the tangent from C meet at P with $AP=t$. Determine all integral values of the pair (d,t) and show that for each one PC is also integral.

No. 242. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries.

For how many positive integral values of B is $R=\sqrt{(B^2+N)}$ an integer, if $N=10395=1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11$? What are the two largest values of R ?

No. 243. Proposed by *Robert C. Yates*, University of Maryland.

Given a straightedge with two marks on it. With this instrument alone construct a rectangular network of lines.

The Golden Section is that part x of a line of unit length such that

$$1/x=x/(1-x)$$

Any odd number sufficiently large can be written as the sum of 3 primes!

Reviews and Abstracts

Edited by
P. K. SMITH and H. A. SIMMONS

Modern Theories of Integration. By H. Kestelman, Oxford University Press, Oxford, 1937. viii+252 pages.

This text is designed to serve as an introduction to the theory of the Lebesgue integral for those who have covered the equivalent of Hardy's *Course of Pure Mathematics.* A discussion of point set theory, Jordan content and Riemann integration leads up to the theory of Lebesgue measure. The rest of the book (a little more than half) is devoted to Lebesgue integrals, the Hölder-Lebesgue integral, the Denjoy integral, and applications to Fourier series.

The theorems and definitions are numbered consecutively and numerous cross references are given in the proofs. The definitions and symbols (71) are tabulated at the end of the book in convenient form. A total of 315 theorems are stated and proved.

The treatment, which is along the lines of Caratheodory's *Vorlesungen über reelle Funktionen,* stresses the relationship between the theory of measure and the theory of integration. The advantages of the Lebesgue theory over that of the Riemann theory for certain types of problems is brought out neatly by a few well chosen examples.

The book should be in the hands of every graduate student interested in real variables. The exposition is clear and the arrangement and printing of the book are good. Only two trivial errors were noted outside of the few listed in the short table of errata.

Virginia Military Institute. W. E. BYRNE.

Plane Trigonometry. By C. N. Mills, Edith Atkin, and Elinor Flagg, Scott, Foresman and Company, New York, 1937. xi+170 pages.

"Easy" is the keynote of this new trigonometry text. The large type gives it the impression of being elementary. "Easy transition to new ideas" is the first of the features noted by the authors. Feature number nine is "easy approach to logarithms". The thirteen-page introductory chapter is devoted to a simple treatment of

scale drawings and similar triangles as a means of determining unknown distances.

Chapter two associates an acute angle with a right triangle placed on a coordinate system and defines the tangent function immediately in terms of coordinates of a point. A list of exercises follows the definition. The same procedure is then used to acquaint the student with the sine function. This is followed by the general definitions of the six functions for any angle and immediately specialized for acute angles using the familiar terms "opposite side", "adjacent side" and "hypotenuse".

The place of trigonometry in the scheme of historical development of mathematics is presented in a worthwhile way on separate pages throughout the text. A total of approximately nine pages is used in this way.

Tables occupy only four pages. These include four-place tables of (a) natural functions from 0° to 90° at intervals of one degree, (b) common logarithms of numbers from 100 to 1000, (c) logarithms of functions from 0° to 90° at intervals of one degree.

In general the number and difficulty of the problems is sufficient for the type of text. Figures are given for an unusually large number of exercises. Answers are not included in the text but can be obtained from the publisher.

Belhaven College. DOROTHY McCOY.

General Mathematics for Students of Business. By William S. Schlauch, F. S. Crofts and Co., New York City, 1936. viii+393 pages.

The design and scope of this book may best be sensed by an examination of the titles of its twenty-four chapters. These titles in the order of their appearance, together with the number of pages in each respective chapter, are as follows:

I The Real Number system and the Fundamental Operations, (18);
II Composition, Factoring, and Fractions, (10);
III Linear Equations and Problems, (16);
IV Simultaneous Equations and Determinants, (8);
V Simultaneous Equations in Problems Arising in Business, (9);
VI Graphs, (11);
VII Theory of Exponents and Logarithms, (11);
VIII Radicals, Imaginaries, and Complex Numbers, (10);

The book contains material for a four-hour-per-week course for 30 weeks. The author also points out chapters and paragraphs for a two-hour-per-week course; and then, similarly, for a three-hour-per-week course. Thus the needs of all parties concerned may be met.

The great aim of the text is to fit students for "Successful work in the mathematical theory of investments, finance, insurance, business calculations, statistics, and budgeting,"... "A text designed to give a *mastery of the mathematical technique and processes* which should be second nature to the student of such technical courses." (The italics are the reviewer's).

The author calls attention to the practical consideration dominating the selection of material, and then further emphasizes these points: The inductive procedure, practical applications of theory, the number of problems taken from C. P. A. examinations, the insight into economic problems, the treatment of curve fitting and trend lines by the Method of Least Squares, and the treatment of forecasting problems.

The prospective teacher will most probably wonder whether the amount of material devoted to algebra, to trigonometry, and to analytical geometry is sufficient to furnish the foundation necessary to the large and, in many places, heavy superstructure of the more advanced and technical matter.

In some instances, the careful mathematician will object to the lack of rigor of certain proofs of formulae. For example, on page 215, in proving that $d(\sin x) = \cos x dx$, if OA, the radius of the circle, is unity, how does it happen that "the sine would have grown to JM", a quantity much larger than OA, (unity), the maximum value of the sine of any angle? Other features of the proof are such that a student can scarcely be expected to follow its details unaided by the teacher.

The present reviewer feels that the author has set for himself a very ambitious program, one exceedingly difficult to carry out successfully, except perhaps with mature or upper-class students.

But, in refutation of these doubts, the author reminds his readers that his work is "the result of years of experience in teaching the subjects treated".

The pages of the book are rather large, and the type used for exercises rather small; the number of errors in printing are few; the binding and general appearance are attractive. The first approach to the book is alluring—it appeals very much to the prospective teacher. Certainly, it is entitled to a try-out by a large number of institutions.

Louisiana State University. IRBY C. NICHOLS.

LITERATURE RECEIVED BY THE EDITORIAL BOARD

1. *Grosse Mathematiker,* By Gerhard Kowalewski. J. F. Lehmann's Verlag, Munich and Berlin, 1938. 300 pages, with 35 figures and 16 portraits. Paper backs, 7.65 marks; linen, 8.70 marks.

2. *Brook Taylor, der Mathematiker und Philosoph.* By Heinrich Auchter. Verlag Konrad Triltsch, Würzburg, 1937. 112 pages, one portrait.

3. *Principles of Mathematics.* By Bertrand Russell. W. W. Norton & Co., Inc. Second edition, New York, 1938. 534 pages. $5.

4. *A Catalogue of a Special Exhibition of Manuscripts, Books, Portraits and Personal Relics of Nathaniel Bowditch* (1773-1838). With a Sketch of the life of Nathaniel Bowditch by Dr. Harold Bowditch and an Essay on the scientific achievements of Nathaniel Bowditch, with a Bibliography of his publications, by Professor Raymond Clare Archibald. (Peabody Museum, Salem, Mass.). Southworth-Anthoensen Press, Portland, Maine. 1937. *iv*+40 pages, 7 portraits.

5. *Csernák László* (Life and Works of the Hungarian Mathematician). By József Jelitai. Különlenyomat a Debreceni Szemle 1937. július—szeptemberi számából. Budapest. 1937, one portrait.

6. *Comptes Rendus du Congrès International des Mathématiciens Oslo* 1936:

Tome I, *Procès-Verbaux et Conférences Générales*, A. W. Brøggers Boktrykkeri A/S. Oslo, 1937. 316 pages.

Tome II, *Conférences de Sections*, A. W. Brøggers Boktrykkeri, A/S, Oslo, 1937. 289 pages.

7. *Erster Bericht der Bernoulli-Kommission.* By Otto Spiess. Buchdruckerei E. Birkhäuser & Cie. Basel, 1937.

8. *Ergänzungen zur Geschichte der Mathematik an der Universität Giessen.* By Wilhelm Lorey. Druck und Verlag: Brühl'sche Univ. = Buch = und Steindruckerei, R. Lange. Giessen. 1937. (in the series *Nachrichten der Giessener Hochschulgesellchaft*, vol. 11, no. 3, ed. by Alfred Götze).

9. *Die mathematischen Leistungen der Semiten im Altertum und Mittelalter.* By Kurt Vogel. Zeitschrift für die gesamte Naturwissenschaft, Heft 2/3, 1937. Friedrich Vieweg & Sohn. Brunswick, 1937.

10. *Wann beginnt die Algebra?* By Kurt Vogel. Semester-Berichte, Mathematisches Seminar Münster in Westfalen. Münster, 1937.

11. *Aus der mathematischen Vergangenheit Münsters.* By Wilhelm Lorey. Semester-Berichte, math. Seminar Münster i. Westf. Münster, 1937.

12. *Versicherungswissenschaftliche Studien an der Universität Göttingen im 18. Jahrhundert.* By Wilhelm Lorey. Zeitschrift für die gesamte Versicherungs-Wissenschaft, vol. 37, no. 3.

13. *The Mathematics of Finance.* By Paul R. Rider. Farrar & Rinehart, Inc., New York, 1938.

14. *College Algebra* (Revised edition). By William L. Hart. D. C. Heath and Company, New York, 1938. Price $2.24.

15. *Pure Mathematics* (Revised edition). By G. H. Hardy. Cambridge University Press, New York, 1938.

16. *Arithmetic for Teacher-Training Classes* (Revised edition). By E. H. Taylor. Henry Holt and Company, New York, 1937.

17. *First Year College Mathematics.* By M. A. Hill, Jr. and J. Burton Linker. Henry Holt and Company, New York, 1936.

18. *Convergence.* By W. L. Farrar. Oxford University Press, New York, 1938.

19. *Analytic Geometry.* By William Wilder Burton. Harcourt Brace and Company, New York, 1938.

20. *Practical Business Mathematics.* By Justin H. Moore and Julio A. Mira. Longmans, Green and Company, New York, 1938.

21. *Freshman Mathematics* (Revised edition). By Hermon L. Slobin and Walter E. Wilbur. Farrar & Rinehart, Inc., New York, 1938.

22. *New Analytic Geometry* (Alternate edition). By Percey F. Smith, Arthur Sullivan Gale, and John Haven Neelley. Ginn and Company, New York, 1938.

23. *Trigonometry* (Revised edition). By A. R. Crathorne and E. B. Lytle. Henry Holt and Company, New York, 1938.

24. *Calculus.* By Edward S. Smith, Meyer Salkover, and Howard K. Justice. John Wiley and Sons, New York, 1938.

25. *Teaching Arithmetic in the Elementary School.* By Robert Lee Morton. Silver Burdett Company, New York, 1937.

26. *Computation and Trigonometry.* By Harold J. Gay. The Macmillan Company, New York, 1938.

27. *Analytic Geometry.* By John Wesley Young, Tomlinson Fort, and Frank Millett Morgan. Houghton Mifflin Co., New York, 1936.

28. *Plane Trigonometry.* By William Kelso Morrill. Farrar & Rinehart, Inc., New York, 1938.

29. *Plane Trigonometry.* By H. .A Simmons and G. D. Gore. John Wiley & Sons, Inc. New York, 1937.

30. *Construction, Classification and Census of Magic Squares of an Even Order.* By Albert L. Candy. Edwards Bros., Inc., Ann Arbor, 1937.

31. *Arithmetic of the Alternating.* By Robert Ashby Philip. The Monographic Press, Fairhaven, Mass., 1937.

32. *Probability Theory and Extra-Sensory Perception.* By H. Rogosin. Reprint from *The Journal of Psychology.* 1938.

33. *Mathematics—The Subtle Fine Art.* By James B. Shaw. Reprint from *The Scientific Monthly,* November, 1934.

34. *On Finite Systems of Linear Differential Equations of Infinite Order with Constant Coefficients.* By Gaines B. Lang. Reprint from *Annals of Mathematics,* January, 1938.

35. *Numerical Identities.* By Alfred Möessner. Reprint from *The Mathematics Student.* Volume 5.

36. *Die Gleichung* $A_1^n+A_2^n+\ldots\ldots+A_z^n=B_1^n+B_2^n+\ldots\ldots+B_z^n$, $n=1,2,\ldots\ldots,8$, *und verwandte Formen.* By Alfred Möessner. Reprint from *The Tôhoku Mathematical Journal,* August, 1935.

37. *Irrational Rhythms.* By James B. Shaw. Reprint from *The Scientific Monthly,* August, 1936.

38. *The Quest of Beauty*. By James B. Shaw. Reprint from *The Open Court*, September, 1931.

39. *The Unity of Mathematics*. By James B. Shaw. Reprint from *The Scientific Monthly*, November, 1937.

40. *The Characterisitic Roots of a Matrix*. By W. V. Parker. Reprint from *Duke Mathematical Journal*, September, 1937.

41. *On Symmetric Determinants*. By W. V. Parker. Reprint from *Bulletin of the American Mathematical Society*, October, 1937.

42. *A Classic Problem in Euclidean Geometry*. By Archibald Henderson. Reprint from *The Journal of the Elisha Mitchell Scientific Society*, December, 1937.

Members of the Editorial Board of this journal acknowledge with appreciation the assistance of the following mathematicians, who at various times have acted as referees for papers submitted for publication in National Mathematics Magazine:

Professor N. Altshiller-Court, University of Oklahoma; Professor Mayme I. Logsdon, University of Chicago; Professor A. E. Landry, Catholic University of America; Professor H. B. Curry, Pennsylvania State College; Professor J. R. Musselman, Western Reserve University; Dr. Arthur Ollivier, Mississippi State College; Mr. W. E. Cox, Mississippi State College; Professor E. H. Umberger, University of Maryland; Professor H. P. Thielman, College of St. Thomas; Professor E. A. Gaumnitz, College of St. Thomas; Dr. C. Grace Shover, Carleton College; Professor O. E. Brown, Case School of Applied Science; Professor E. D. Jenkins, Eastern Kentucky State Teachers College; Professor C. J. Blackall, College of St. Thomas.

Waring's famous problem: "How many kth powers are needed to represent any number?" was answered after 170 years by Hilbert in 1910.

SIR ISAAC NEWTON AND JOHN LOCKE

These two good friends, and two of Britain's most famous scientists, Newton a mathematician, and physicist, and Locke a great philosopher, were walking along the English country side one day, when Newton, as was his frequent custom, began working mentally on a problem which was interesting him. After carrying out the investigation he obtained a result, and turning to his friend, Locke, he said, "And that gives us x". Just to carry on conversation, and appear interested, Locke replied, "Does it"? Newton, surprised and worried, cried, "Doesn't it?" At once he plunged into the solution again, found that he had made a mistake, and said to Locke, "You were right, it does not give us x, it gives us y". Forever after, Sir Isaac Newton regarded John Locke the most erudite of philosophers, never dreaming that Locke had no idea of the problem which Newton had solved.

Teacher: Make a three letter word with the letters a,b,c,d.
Student: B-A-D.
Teacher: That's Good!

Euclid proved that the set of prime numbers was infinite.

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PROBLEMS PROPOSED

Of each pair of successive numbers followed by a semicolon, or a period, the first denotes the NUMBER of the problem proposed, the second, the PAGE on which it is stated.

PROBLEMS SOLVED

Of each pair of successive numbers followed by a semicolon, or a period, the first denotes the NUMBER of the problem proposed, the second, the PAGE on which it is solved.

THE TERCENTENARY OF THE BIRTH OF

James Gregory

will be celebrated at a Mathematical Colloquium to be held in St. Andrews, Scotland, from July 4th to 15th, 1938, under the auspices of the Edinburgh Mathematical Society. Courses of lectures have been arranged on subjects of Pure Mathematics, Mathematical Physics and Mathematical Biology. The list of lecturers includes A. C. Aitken, G. D. Birkhoff, W. O. Kermack, H. W. Turnbull and E. T. Whittaker.

Members of the Colloquium will have the opportunity of seeing the many mediaeval buildings of great historical interest in St. Andrews, and of playing on the famous golf courses. By courtesy of the St. Andrews University Court, the Colloquium will be held in the University Hall, where members of the Colloquium may stay with their families. Full particulars of membership of the Colloquium (which is not confined to members of the Society) may be had from the Hon. Secretary, Edinburgh Mathematical Society, 16 Chambers Street, Edinburgh.

(*Communication from J. M. N. Etherington, Hon. Secretary Edinburgh Mathematical Society.*)